

# SOLUTIONS OF WEINSTEIN EQUATIONS REPRESENTABLE BY BESSEL POISSON INTEGRALS OF BMO FUNCTIONS

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**ABSTRACT.** We consider the Weinstein type equation  $\mathcal{L}_\lambda u = 0$  on  $(0, \infty) \times (0, \infty)$ , where  $\mathcal{L}_\lambda = \partial_t^2 + \partial_x^2 - \frac{\lambda(\lambda-1)}{x^2}$ , with  $\lambda > 1$ . In this paper we characterize the solutions of  $\mathcal{L}_\lambda u = 0$  on  $(0, \infty) \times (0, \infty)$  representable by Bessel-Poisson integrals of  $BMO$ -functions as those ones satisfying certain Carleson properties.

## 1. INTRODUCTION

The space  $BMO(\mathbb{R}^n)$  of bounded mean oscillation functions in  $\mathbb{R}^n$  was introduced by John and Nirenberg ([37]) in the context of partial differential equations. A function  $f \in L^1_{\text{loc}}(\mathbb{R}^n)$  is in  $BMO(\mathbb{R}^n)$  provided that

$$\|f\|_{BMO(\mathbb{R}^n)} := \sup_B \frac{1}{|B|} \int_B |f(x) - f_B| dx < \infty,$$

where the supremum is taken over all balls  $B$  in  $\mathbb{R}^n$ . Here,  $|B|$  denotes the Lebesgue measure of  $B$  and  $f_B$  represents the average of  $f$  on  $B$ , that is,  $f_B = \frac{1}{|B|} \int_B f(x) dx$ . By identifying those functions that differ by a constant,  $(BMO(\mathbb{R}^n), \|\cdot\|_{BMO(\mathbb{R}^n)})$  is a Banach space.

A celebrated result of Fefferman and Stein ([31]) establishes that  $BMO(\mathbb{R}^n)$  is the dual space of the Hardy space  $H^1(\mathbb{R}^n)$ . The spaces  $H^1(\mathbb{R}^n)$  and  $BMO(\mathbb{R}^n)$  turned out to be the correct substitutes for  $L^1(\mathbb{R}^n)$  and  $L^\infty(\mathbb{R}^n)$ , respectively, as the domain and the target spaces of operators appearing in harmonic analysis.

Since Fefferman and Stein's paper ([31]) appeared, the space of bounded mean oscillation functions has motivated the investigations of many mathematicians (see, for instance, [15], [18], [19], [21], [22], [32], [35], [36], [38], [41], [43], [47], [48], [52] and [53]).

The space  $BMO(\mathbb{R}^n)$  is closely connected to certain positive measures in  $\mathbb{R}^{n+1}_+$  known as Carleson measures. These measures were introduced by Carleson to solve the corona problem ([13]). A positive measure  $\mu$  on  $\mathbb{R}^{n+1}_+$  is called a Carleson measure when

$$(1) \quad \|\mu\|_{\mathcal{C}} := \sup_Q \frac{\mu(Q \times (0, \ell(Q)))}{|Q|} < \infty,$$

where the supremum is taken over all cubes  $Q$  in  $\mathbb{R}^n$ . Here  $\ell(Q)$  denotes the length of the edge of  $Q$ .

If  $f$  is a measurable function on  $\mathbb{R}^n$  such that  $\int_{\mathbb{R}^n} |f(x)|(1 + |x|)^{-n-1} dx < \infty$ , then, for every  $t > 0$ , the Poisson integral  $P_t(f)$  of  $f$  is defined by

$$P_t(f)(x) = \int_{\mathbb{R}^n} P_t(x - y) f(y) dy, \quad x \in \mathbb{R}^n \text{ and } t > 0,$$

where

$$P_t(z) = \frac{\Gamma(n + 1/2)}{\pi^{n+1/2}} \frac{t}{(|z|^2 + t^2)^{(n+1)/2}}, \quad z \in \mathbb{R}^n \text{ and } t > 0.$$

The characterization of the bounded mean oscillation functions via Carleson measures was given by Fefferman and Stein.

**Theorem A.** ([31, p. 145]) *Let  $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ . Then,  $f \in BMO(\mathbb{R}^n)$  if, and only if,  $\int_{\mathbb{R}^n} |f(x)|(1 + |x|)^{-n-1} dx < \infty$  and the measure  $t|\nabla P_t(f)(x)|^2 dx dt$  is Carleson in  $\mathbb{R}^{n+1}_+$ , where  $\nabla = (\partial_{x_1}, \dots, \partial_{x_n}, \partial_t)$ .*

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Some versions of this result for  $BMO$ -type spaces associated with operators have been established in the last decade (see [6], [23], [25], [35] and [40], amongst others).

Theorem A was completed by Fabes, Johnson and Neri ([28] and [29]). An harmonic function  $u$  defined on  $\mathbb{R}_+^{n+1}$  is said to be in  $HMO(\mathbb{R}_+^{n+1})$  provided that

$$\sup_Q \frac{1}{|Q|} \int_0^{\ell(Q)} \int_Q |t \nabla u(x, t)|^2 \frac{dx dt}{t} < \infty,$$

where the supremum is taken over all cubes in  $\mathbb{R}^n$ . Theorem A implies that  $P_t(BMO(\mathbb{R}^n)) \subseteq HMO(\mathbb{R}_+^{n+1})$  suitably understood. The equality is established in the following.

**Theorem B.** ([29, Theorem 1.0]). *A function  $u \in HMO(\mathbb{R}_+^{n+1})$  if, and only if, there exists  $f \in BMO(\mathbb{R}^n)$  such that  $u(x, t) = P_t(f)(x)$ ,  $(x, t) \in \mathbb{R}_+^{n+1}$ .*

Other proof of Theorem B can be encountered in [28, Theorem 1.0]. The result in Theorem B was also proved by Fabes and Neri [30] when  $\mathbb{R}_+^{n+1}$  is replaced by a bounded starlike Lipschitz domain ([30, Theorem p. 35]). More recently, Chen [16, Theorems A and 18] and Chen and Luo [17] characterized the harmonic functions in  $M \times (0, \infty)$ , where  $M$  is a complete connected Riemannian manifold with positive Ricci curvature, having  $BMO(M)$  traces. In [23, Theorem 1.1] Duong, Yang, and Zhang have extended Theorem B to Poisson integrals and  $BMO$  functions in the Schrödinger operator setting.

Our objective in this paper is to establish a version of Theorem B in the Bessel operator context.

The study of harmonic analysis associated with Bessel operators was began in a systematic way by Muckenhoupt and Stein ([42]). In the last decade Bessel harmonic analysis has been developed (see, for instance, [3], [4], [8], [9], [10], [24], [45] and [54]).

For  $\lambda > 0$  we consider the Bessel operator on  $(0, \infty)$

$$B_\lambda = \frac{d^2}{dx^2} - \frac{\lambda(\lambda-1)}{x^2} = x^{-\lambda} \frac{d}{dx} x^{2\lambda} \frac{d}{dx} x^{-\lambda}.$$

According to [42, §16] the Poisson semigroup  $\{P_t^\lambda\}_{t>0}$  associated with the Bessel operator  $B_\lambda$  is defined as follows. If  $f \in L^p(0, \infty)$ ,  $1 \leq p \leq \infty$ ,

$$P_t^\lambda(f)(x) = \int_0^\infty P_t^\lambda(x, y) f(y) dy, \quad x, t \in (0, \infty),$$

where

$$P_t^\lambda(x, y) = \frac{2\lambda(xy)^\lambda t}{\pi} \int_0^\pi \frac{(\sin \theta)^{2\lambda-1}}{((x-y)^2 + t^2 + 2xy(1 - \cos \theta))^{\lambda+1}} d\theta, \quad x, y, t \in (0, \infty).$$

$L^p$ -boundedness properties of  $\{P_t^\lambda\}_{t>0}$  and the associated maximal operators were studied in [11] and [46]. The semigroup  $\{P_t^\lambda\}_{t>0}$  has not the Markovian property, that is,  $P_t^\lambda$  does not map constants into constants. This fact produces technical difficulties when studying Bessel Poisson semigroups on functions of bounded mean oscillation (see [6]).

We denote by  $BMO_o(\mathbb{R})$  the space of all those odd functions  $f \in BMO(\mathbb{R})$ . Let  $1 < p < \infty$ . For every  $f \in BMO_o(\mathbb{R})$  there exists  $C > 0$  such that,

$$(2) \quad \left\{ \frac{1}{|I|} \int_I |f(x) - f_I|^p dx \right\}^{1/p} \leq C,$$

for every interval  $I = (a, b)$ ,  $0 < a < b < \infty$  and

$$(3) \quad \left\{ \frac{1}{|I|} \int_I |f(x)|^p dx \right\}^{1/p} \leq C,$$

for each interval  $I = (0, b)$ ,  $0 < b < \infty$ . Moreover, the quantity  $\inf\{C > 0 : (2) \text{ and } (3) \text{ hold}\}$  is equivalent to  $\|f\|_{BMO(\mathbb{R})}$ .

Conversely, if  $f \in L_{loc}^1[0, \infty)$  satisfies (2) and (3) for all admissible intervals, then the odd extension  $f_o$  of  $f$  to  $\mathbb{R}$  belongs to  $BMO_o(\mathbb{R})$  and  $\|f_o\|_{BMO(\mathbb{R})}$  is equivalent to the quantity  $\|f\|_{BMO_o(\mathbb{R})} := \inf\{C > 0 : (2) \text{ and } (3) \text{ hold}\}$  (see [12, Proposition 12] and [50, Corollary p. 144]). In this case we also say that  $f \in BMO_o(\mathbb{R})$ .

As in (1) we say that a positive measure  $\mu$  on  $(0, \infty) \times (0, \infty)$  is Carleson when

$$\|\mu\|_{\mathcal{C}} := \sup_I \frac{\mu(I \times (0, |I|))}{|I|} < \infty,$$

where the supremum is taken over all bounded intervals  $I \subset (0, \infty)$ .

In [6] it was proved a Bessel version of Theorem A.

**Theorem C.** ([6, Theorem 1.1]). *Let  $\lambda > 0$ . Assume that  $f \in L^1_{\text{loc}}[0, \infty)$ . Then, the following assertions are equivalent.*

- (i)  $f \in BMO_o(\mathbb{R})$ .
- (ii)  $(1 + x^2)^{-1}f \in L^1(0, \infty)$  and

$$d\gamma_f(x, t) = |t\partial_t P_t^\lambda(f)(x)|^2 \frac{dxdt}{t}$$

*is a Carleson measure on  $(0, \infty) \times (0, \infty)$ .*

Moreover, the quantities  $\|f\|_{BMO_o(\mathbb{R})}^2$  and  $\|\gamma_f\|_{\mathcal{C}}$  are equivalent.

REMARK. Another characterization of  $BMO_o(\mathbb{R})$ , slightly different to (ii) in Theorem C and that will be used in Section 3, is given in Lemma 3.1.

If  $\Omega \subseteq (0, \infty) \times \mathbb{R}$  we say that a function  $u \in C^2(\Omega)$  is  $\lambda$ -harmonic provided that

$$\partial_t^2 u(x, t) + B_{\lambda, x} u(x, t) = 0, \quad (x, t) \in \Omega.$$

The operator  $\partial_t^2 + B_{\lambda, x}$  is related to the Weinstein operator associated with the generalized axially symmetric potential theory (see [14] and the references there). We can write  $B_\lambda = -D_\lambda^* D_\lambda$ , where  $D_\lambda = x^\lambda \frac{d}{dx} x^{-\lambda}$  and  $D_\lambda^*$  is the formal adjoint operator of  $D_\lambda$  in  $L^2(0, \infty)$ . We define the  $\lambda$ -gradient  $\nabla_\lambda$  by

$$\nabla_\lambda = (D_{\lambda, x}, \partial_t).$$

The main result of this paper is the following.

**Theorem 1.** *Let  $\lambda > 1$ . Assume that  $u$  is a  $\lambda$ -harmonic function on  $(0, \infty) \times (0, \infty)$  such that  $x^{-\lambda} u(x, t) \in C^\infty(\mathbb{R} \times (0, \infty))$  and is even in the  $x$ -variable. Then, the following assertions are equivalent.*

- (i) *There exists  $f \in BMO_o(\mathbb{R})$  such that  $u(x, t) = P_t^\lambda(f)(x)$ ,  $(x, t) \in (0, \infty) \times (0, \infty)$ .*
- (ii) *The measure*

$$d\mu_\lambda(x, t) = |t\nabla_\lambda u(x, t)|^2 \frac{dxdt}{t}$$

*is Carleson on  $(0, \infty) \times (0, \infty)$ .*

Moreover, the quantities  $\|f\|_{BMO_o(\mathbb{R})}^2$  and  $\|\mu_\lambda\|_{\mathcal{C}}$  are equivalent.

Note that the property (ii) in Theorem 1 is stronger than the condition (ii) in Theorem C.

In the next sections we prove Theorem 1. In the sequel by  $C$  we always denote a positive constant not necessarily the same in each occurrence.

## 2. PROOF OF (i) $\implies$ (ii) IN THEOREM 1

As it can be observed along the proof, this part of Theorem 1 is valid for  $\lambda > 0$ .

Assume that  $u(x, t) = P_t^\lambda(f)(x)$ ,  $x, t \in (0, \infty)$ , for a certain  $f \in BMO_o(\mathbb{R})$ . According to Theorem C the measure

$$d\gamma_f(x, t) = |t\partial_t P_t^\lambda(f)(x)|^2 \frac{dxdt}{t}$$

is Carleson on  $(0, \infty) \times (0, \infty)$ . Moreover, we have that

$$\|\gamma_f\|_{\mathcal{C}} \leq C \|f\|_{BMO_o(\mathbb{R})}^2,$$

where  $C > 0$  does not depend on  $f$ .

We are going to see that the measure

$$d\rho_f(x, t) = |tD_{\lambda, x} P_t^\lambda(f)(x)|^2 \frac{dxdt}{t}$$

is Carleson on  $(0, \infty) \times (0, \infty)$  and that

$$\|\rho_f\|_{\mathcal{C}} \leq C \|f\|_{BMO_o(\mathbb{R})}^2,$$

for certain  $C > 0$  which does not depend on  $f$ .

Let  $I = (a, b)$  where  $0 \leq a < b < \infty$ . We decompose  $f$  as follows

$$f = (f - f_{2I})\chi_{2I} + (f - f_{2I})\chi_{(0, \infty) \setminus 2I} + f_{2I} =: f_1 + f_2 + f_3.$$

Here  $2I = (x_I - |I|, x_I + |I|) \cap (0, \infty)$  and  $x_I = (a + b)/2$ .

We will prove that

$$(4) \quad \frac{1}{|I|} \int_0^{|I|} \int_I |tD_{\lambda,x}P_t^\lambda(f_j)(x)|^2 \frac{dxdt}{t} \leq C\|f\|_{BMO_o(\mathbb{R})}^2, \quad j = 1, 2, 3,$$

for certain  $C > 0$  independent of  $I$  and  $f$ .

**2.1. Proof of (4) for  $j = 1$ .** We introduce the Littlewood-Paley function  $g_\lambda$  defined by

$$g_\lambda(F)(x) = \left( \int_0^\infty |tD_{\lambda,x}P_t^\lambda(F)(x)|^2 \frac{dt}{t} \right)^{1/2}, \quad x \in (0, \infty),$$

for every  $F \in L^2(0, \infty)$ .

**Lemma 2.1.** *Let  $\lambda > 0$ . The Littlewood-Paley function  $g_\lambda$  is a bounded (sublinear) operator from  $L^2(0, \infty)$  into itself.*

*Proof.* We consider the Hankel transformation  $h_\lambda$  defined by

$$h_\lambda(F)(x) = \int_0^\infty \sqrt{xy} J_{\lambda-1/2}(xy) F(y) dy, \quad x \in (0, \infty),$$

for every  $F \in L^1(0, \infty)$ . Here  $J_\nu$  denotes the Bessel function of the first kind and order  $\nu$ . The transformation  $h_\lambda$  can be extended from  $L^1(0, \infty) \cap L^2(0, \infty)$  to  $L^2(0, \infty)$  as an isometry of  $L^2(0, \infty)$ , where  $h_\lambda^{-1} = h_\lambda$  ([51, Ch. VIII]).

Let  $F \in L^2(0, \infty)$ . According to [42, (16.1')] we have that

$$P_t^\lambda(F)(x) = h_\lambda(e^{-yt} h_\lambda(F))(x), \quad x, t \in (0, \infty).$$

Since  $\frac{d}{dz}(z^{-\nu} J_\nu(z)) = -z^{-\nu} J_{\nu+1}(z)$ ,  $z \in (0, \infty)$ , and  $e^{-yt} h_\lambda(F) \in L^1(0, \infty)$ ,  $t > 0$ , we get

$$D_{\lambda,x}P_t^\lambda(F)(x) = -h_{\lambda+1}(ye^{-yt} h_\lambda(F))(x), \quad x, t \in (0, \infty).$$

Then,

$$\begin{aligned} \|g_\lambda(F)\|_{L^2(0,\infty)}^2 &= \int_0^\infty \int_0^\infty |tD_{\lambda,x}P_t^\lambda(F)(x)|^2 \frac{dxdt}{t} \\ &= \int_0^\infty t \int_0^\infty |h_{\lambda+1}(ye^{-yt} h_\lambda(F)(y))(x)|^2 dxdt \\ &= \int_0^\infty t \int_0^\infty y^2 e^{-2ty} |h_\lambda(F)(y)|^2 dydt \\ &= \int_0^\infty y^2 |h_\lambda(F)(y)|^2 \int_0^\infty t e^{-2ty} dt dy = \frac{1}{4} \|h_\lambda(F)\|_{L^2(0,\infty)}^2 = \frac{1}{4} \|F\|_{L^2(0,\infty)}^2. \end{aligned}$$

□

Lemma 2.1 leads to

$$\begin{aligned} \frac{1}{|I|} \int_0^{|I|} \int_I |tD_{\lambda,x}P_t^\lambda(f_1)(x)|^2 \frac{dxdt}{t} &\leq \frac{1}{|I|} \int_0^\infty |g_\lambda(f_1)(x)|^2 dx \leq \frac{C}{|I|} \int_0^\infty |f_1(y)|^2 dy \\ &= \frac{C}{|I|} \int_{2I} |f(y) - f_{2I}|^2 dy \leq C\|f\|_{BMO_o(\mathbb{R})}^2, \end{aligned}$$

being  $C$  independent of  $I$  and  $f$ .

**2.2. Proof of (4) for  $j = 2$ .** First of all we establish the following estimation for the kernel  $D_{\lambda,x}P_t^\lambda(x, y)$ ,  $x, y, t \in (0, \infty)$ .

**Lemma 2.2.** *Let  $\lambda > 0$ . Then,*

$$|D_{\lambda,x}P_t^\lambda(x, y)| \leq \frac{C}{(x-y)^2 + t^2}, \quad x, y, t \in (0, \infty).$$

*Proof.* We write the following decomposition

$$\begin{aligned} D_{\lambda,x}P_t^\lambda(x, y) &= -\frac{4\lambda(\lambda+1)}{\pi} (xy)^\lambda t \int_0^\pi \frac{(\sin \theta)^{2\lambda-1} (x - y \cos \theta)}{((x-y)^2 + t^2 + 2xy(1 - \cos \theta))^{\lambda+2}} d\theta \\ (5) \quad &=: P_t^{\lambda,1}(x, y) + P_t^{\lambda,2}(x, y), \quad x, y, t \in (0, \infty), \end{aligned}$$

where

$$P_t^{\lambda,1}(x, y) = -\frac{4\lambda(\lambda+1)}{\pi}(xy)^\lambda t \int_0^{\pi/2} \frac{(\sin \theta)^{2\lambda-1}(x-y \cos \theta)}{((x-y)^2 + t^2 + 2xy(1-\cos \theta))^{\lambda+2}} d\theta, \quad x, y, t \in (0, \infty).$$

We have that

$$(6) \quad |P_t^{\lambda,2}(x, y)| \leq C(xy)^\lambda t \frac{x+y}{((x-y)^2 + t^2 + 2xy)^{\lambda+2}} \leq \frac{C}{(x-y)^2 + t^2}, \quad x, y, t \in (0, \infty).$$

On the other hand, since

$$|x - y \cos \theta| \leq |x - y| + \min\{x, y\}(1 - \cos \theta), \quad x, y \in (0, \infty), \quad \theta \in \mathbb{R},$$

and  $\sin \theta \sim \theta$  and  $2(1 - \cos \theta) \sim \theta^2$ ,  $\theta \in [0, \pi/2]$ , it follows that

$$|P_t^{\lambda,1}(x, y)| \leq C(P_t^{\lambda,1,1}(x, y) + P_t^{\lambda,1,2}(x, y)), \quad x, y, t \in (0, \infty),$$

where

$$P_t^{\lambda,1,1}(x, y) = (xy)^\lambda t |x - y| \int_0^{\pi/2} \frac{\theta^{2\lambda-1}}{((x-y)^2 + t^2 + xy\theta^2)^{\lambda+2}} d\theta, \quad x, y, t \in (0, \infty),$$

and

$$P_t^{\lambda,1,2}(x, y) = (xy)^\lambda t \min\{x, y\} \int_0^{\pi/2} \frac{\theta^{2\lambda+1}}{((x-y)^2 + t^2 + xy\theta^2)^{\lambda+2}} d\theta, \quad x, y, t \in (0, \infty).$$

We get

$$\begin{aligned} P_t^{\lambda,1,1}(x, y) &\leq C(xy)^\lambda t |x - y| \int_0^\infty \frac{\theta^{2\lambda-1}}{((x-y)^2 + t^2 + xy\theta^2)^{\lambda+2}} d\theta \\ &\leq C \frac{t|x-y|}{((x-y)^2 + t^2)^2} \leq \frac{C}{(x-y)^2 + t^2}, \quad x, y, t \in (0, \infty), \end{aligned}$$

and

$$\begin{aligned} P_t^{\lambda,1,2}(x, y) &\leq C \frac{(xy)^\lambda t \min\{x, y\}}{((x-y)^2 + t^2)^{3/2}} \int_0^{\pi/2} \frac{\theta^{2\lambda+1}}{(xy\theta^2)^{\lambda+1/2}} d\theta \\ &\leq \frac{C}{(x-y)^2 + t^2} \frac{\min\{x, y\}}{\sqrt{xy}} \leq \frac{C}{(x-y)^2 + t^2}, \quad x, y, t \in (0, \infty). \end{aligned}$$

Hence,

$$(7) \quad |P_t^{\lambda,1}(x, y)| \leq \frac{C}{(x-y)^2 + t^2}, \quad x, y, t \in (0, \infty).$$

Thus, the result follows from (5), (6) and (7).  $\square$

We now proceed as in [6, pp. 468-469]. By Lemma 2.2 we can write

$$\begin{aligned} |D_{\lambda,x} P_t^\lambda(f_2)(x)| &\leq C \int_{(0,\infty) \setminus 2I} \frac{|f(y) - f_{2I}|}{(x-y)^2 + t^2} dy \leq C \int_{(0,\infty) \setminus 2I} \frac{|f(y) - f_{2I}|}{(x_I - y)^2 + t^2} dy \\ &\leq \frac{C}{|I|} \sum_{k=1}^\infty \frac{1}{2^k} \left( \frac{1}{2^k |I|} \int_{2^{k+1}I} |f(y) - f_{2^{k+1}I}| dy + |f_{2^{k+1}I} - f_{2I}| \right) \\ &\leq \frac{C}{|I|} \|f\|_{BMO_o(\mathbb{R})}, \quad x \in I \text{ and } t > 0. \end{aligned}$$

In the last inequality we have taken into account [33, Ch. VI (1.3)] and that, if  $k \in \mathbb{N} \setminus \{0\}$  and  $2^k |I| > x_I$ , then  $2^{k+1}I \subset (0, 2^{k+1}|I|)$  and

$$\int_{2^{k+1}I} |f(y) - f_{2^{k+1}I}| dy \leq \int_0^{2^{k+1}|I|} (|f(y)| + |f_{2^{k+1}I}|) dy \leq 2^{k+3}|I| \|f\|_{BMO_o(\mathbb{R})}.$$

We conclude that

$$\frac{1}{|I|} \int_0^{|I|} \int_I |t D_{\lambda,x} P_t^\lambda(f_2)(x)|^2 \frac{dx dt}{t} \leq C \|f\|_{BMO_o(\mathbb{R})}^2,$$

with  $C$  independent of  $I$  and  $f$ .

**2.3. Proof of (4) for  $j = 3$ .** Note firstly that

$$|f_{2I}| \leq \frac{1}{|I|} \int_{2I} |f(y)| dy \leq \frac{x_I + |I|}{|I|} \|f\|_{BMO_o(\mathbb{R})}.$$

Then, estimation (4) for  $j = 3$  will be proved once we show the following.

**Lemma 2.3.** *Let  $\lambda > 0$ . There exists  $C > 0$  such that*

$$(8) \quad \frac{(x_J + |J|)^2}{|J|^3} \int_0^{|J|} \int_J |t D_{\lambda, x} P_t^\lambda(1)(x)|^2 \frac{dx dt}{t} \leq C,$$

for every bounded interval  $J$  on  $(0, \infty)$ .

*Proof.* We take in mind the decomposition (5). As in (6) we get

$$|P_t^{\lambda, 2}(x, y)| \leq C \frac{(x + y)t}{(x^2 + y^2 + t^2)^2} \leq \frac{C}{x^2 + y^2 + t^2}, \quad x, y, t \in (0, \infty).$$

Then,

$$(9) \quad \left| \int_0^\infty P_t^{\lambda, 2}(x, y) dy \right| \leq C \int_0^\infty \frac{dy}{x^2 + y^2 + t^2} \leq \frac{C}{x + t}, \quad x, t \in (0, \infty).$$

Now we write the following splitting

$$\begin{aligned} \int_0^\infty P_t^{\lambda, 1}(x, y) dy &= \left( \int_0^{x/2} + \int_{x/2}^{3x/2} + \int_{3x/2}^\infty \right) P_t^{\lambda, 1}(x, y) dy \\ &=: Q_t^1(x) + Q_t^2(x) + Q_t^3(x), \quad x \in (0, \infty). \end{aligned}$$

According to (7) we get

$$|Q_t^1(x)| \leq C \int_0^{x/2} \frac{dy}{(x - y)^2 + t^2} \leq C \frac{x}{x^2 + t^2} \leq \frac{C}{x + t}, \quad x, t \in (0, \infty),$$

and

$$|Q_t^3(x)| \leq C \int_{3x/2}^\infty \frac{dy}{(y - x + t)^2} \leq \frac{C}{x + t}, \quad x, t \in (0, \infty).$$

We decompose  $Q_t^2(x)$ ,  $t, x \in (0, \infty)$ , in the following way.

$$\begin{aligned} Q_t^2(x) &= -\frac{4\lambda(\lambda + 1)}{\pi} t \int_{x/2}^{3x/2} (xy)^\lambda y \int_0^{\pi/2} \frac{(1 - \cos \theta)(\sin \theta)^{2\lambda-1}}{((x - y)^2 + t^2 + 2xy(1 - \cos \theta))^{\lambda+2}} d\theta dy \\ &\quad - \frac{4\lambda(\lambda + 1)}{\pi} t \int_{x/2}^{3x/2} (xy)^\lambda (x - y) \\ &\quad \times \int_0^{\pi/2} \left( \frac{(\sin \theta)^{2\lambda-1}}{((x - y)^2 + t^2 + 2xy(1 - \cos \theta))^{\lambda+2}} - \frac{\theta^{2\lambda-1}}{((x - y)^2 + t^2 + xy\theta^2)^{\lambda+2}} \right) d\theta dy \\ &\quad + \frac{4\lambda(\lambda + 1)}{\pi} t \int_{x/2}^{3x/2} (xy)^\lambda (x - y) \int_{\pi/2}^\infty \frac{\theta^{2\lambda-1}}{((x - y)^2 + t^2 + xy\theta^2)^{\lambda+2}} d\theta dy \\ &\quad - \frac{4\lambda(\lambda + 1)}{\pi} t \int_{x/2}^{3x/2} (xy)^\lambda (x - y) \int_0^\infty \frac{\theta^{2\lambda-1}}{((x - y)^2 + t^2 + xy\theta^2)^{\lambda+2}} d\theta dy \\ &=: \sum_{j=1}^4 I_j(x, t), \quad x, t \in (0, \infty). \end{aligned}$$

Observe firstly that  $I_4(x, t) = 0$ ,  $t, x \in (0, \infty)$ . Indeed, we have that

$$\begin{aligned} I_4(x, t) &= -\frac{4\lambda(\lambda + 1)}{\pi} t \int_0^\infty \frac{u^{2\lambda-1}}{(1 + u^2)^{\lambda+2}} du \int_{x/2}^{3x/2} \frac{(xy)^\lambda (x - y)}{(xy)^\lambda ((x - y)^2 + t^2)^2} dy \\ &= -\frac{2t}{\pi} \int_{-x/2}^{x/2} \frac{z}{(z^2 + t^2)^2} dz = 0, \quad x, t \in (0, \infty). \end{aligned}$$

We are going to see that

$$|I_j(x, t)| \leq \frac{C}{x + t}, \quad x, t \in (0, \infty) \text{ and } j = 1, 2, 3.$$

Since  $2(1 - \cos \theta) \sim \theta^2$  and  $\sin \theta \sim \theta$ , when  $\theta \in [0, \pi/2]$ , we can write

$$\begin{aligned}
|I_1(x, t)| &\leq Ctx^{2\lambda+1} \int_{x/2}^{3x/2} \int_0^{\pi/2} \frac{\theta^{2\lambda+1}}{((x-y)^2 + t^2 + (x\theta)^2)^{\lambda+2}} d\theta dy \\
&\leq Ct \int_{x/2}^{3x/2} \int_0^\infty \frac{d\theta dy}{((x-y)^2 + t^2 + (x\theta)^2)^{3/2}} \\
&\leq C \frac{t}{x} \int_{x/2}^{3x/2} \frac{dy}{(x-y)^2 + t^2} \leq C \frac{t}{x} \int_0^{x/2} \frac{dz}{z^2 + t^2} \\
&\leq C \frac{t}{x} \int_0^{x/2} \frac{dz}{(z+t)^2} \leq \frac{C}{x+t}, \quad x, t \in (0, \infty).
\end{aligned}$$

Also,

$$\begin{aligned}
|I_3(x, t)| &\leq Ctx^{2\lambda} \int_{x/2}^{3x/2} \int_{\pi/2}^\infty \frac{\theta^{2\lambda-1}}{((x-y)^2 + t^2 + (x\theta)^2)^{\lambda+3/2}} d\theta dy \\
&\leq Ct \int_{x/2}^{3x/2} \frac{1}{((x-y)^2 + t^2)^{3/2}} \int_{\frac{\pi}{2}}^\infty \frac{u^{2\lambda-1}}{(1+u^2)^{\lambda+3/2}} du dy \\
&\leq C \frac{t}{x} \int_{x/2}^{3x/2} \frac{\sqrt{(x-y)^2 + t^2}}{((x-y)^2 + t^2)^{3/2}} dy \int_0^\infty \frac{u^{2\lambda}}{(1+u^2)^{\lambda+3/2}} du \\
&\leq C \frac{t}{x} \int_{x/2}^{3x/2} \frac{dy}{(x-y)^2 + t^2} \leq \frac{C}{x+t}, \quad x, t \in (0, \infty).
\end{aligned}$$

By using that  $|(\sin \theta)^{2\lambda-1} - \theta^{2\lambda-1}| \leq C\theta^{2\lambda+1}$ ,  $\theta \in (0, \pi/2)$ , and that

$$\left| \frac{1}{((x-y)^2 + t^2 + 2xy(1 - \cos \theta))^{\lambda+2}} - \frac{1}{((x-y)^2 + t^2 + xy\theta^2)^{\lambda+2}} \right| \leq C \frac{xy\theta^4}{((x-y)^2 + t^2 + xy\theta^2)^{\lambda+3}},$$

for each  $\theta \in (0, \pi/2)$  and  $t, x, y \in (0, \infty)$ , we obtain

$$\begin{aligned}
&\left| \frac{(\sin \theta)^{2\lambda-1}}{((x-y)^2 + t^2 + 2xy(1 - \cos \theta))^{\lambda+2}} - \frac{\theta^{2\lambda-1}}{((x-y)^2 + t^2 + xy\theta^2)^{\lambda+2}} \right| \\
&\leq C \left( \frac{\theta^{2\lambda+1}}{((x-y)^2 + t^2 + 2xy(1 - \cos \theta))^{\lambda+2}} + \theta^{2\lambda-1} \frac{xy\theta^4}{((x-y)^2 + t^2 + xy\theta^2)^{\lambda+3}} \right) \\
&\leq C \frac{\theta^{2\lambda+1}}{((x-y)^2 + t^2 + xy\theta^2)^{\lambda+2}}, \quad \theta \in \left(0, \frac{\pi}{2}\right), \quad x, y, t \in (0, \infty).
\end{aligned}$$

Then,

$$\begin{aligned}
|I_2(x, t)| &\leq Ctx^{2\lambda} \int_{x/2}^{3x/2} |x-y| \int_0^{\pi/2} \frac{\theta^{2\lambda+1}}{((x-y)^2 + t^2 + (x\theta)^2)^{\lambda+2}} d\theta dy \\
&\leq Ctx^{2\lambda} \int_{x/2}^{3x/2} \int_0^{\pi/2} \frac{\theta^{2\lambda+1}}{((x-y)^2 + t^2 + (x\theta)^2)^{\lambda+3/2}} d\theta dy \\
&\leq Ctx^{2\lambda} \int_{x/2}^{3x/2} \frac{dy}{(x-y)^2 + t^2} \int_0^{\pi/2} \frac{\theta^{2\lambda+1}}{(x\theta)^{2\lambda+1}} d\theta \\
&\leq C \frac{t}{x} \int_{x/2}^{3x/2} \frac{dy}{(x-y)^2 + t^2} \leq \frac{C}{x+t}, \quad x, t \in (0, \infty).
\end{aligned}$$

We conclude that

$$|Q_t^2(x)| \leq \frac{C}{x+t}, \quad x, t \in (0, \infty).$$

Hence,

$$(10) \quad \left| \int_0^\infty P_t^{\lambda,1}(x, y) dy \right| \leq \frac{C}{x+t}, \quad x, t \in (0, \infty).$$

Let  $J$  a bounded interval in  $(0, \infty)$ . If  $x_J < |J|$ , we obtain by (9) and (10)

$$\frac{(x_J + |J|)^2}{|J|^3} \int_0^{|J|} \int_J t \left| \int_0^\infty |D_{\lambda,x} P_t^\lambda(x, y) dy| \right|^2 dx dt \leq \frac{C}{|J|} \int_0^{|J|} \int_0^{2|J|} \frac{t}{(x+t)^2} dx dt$$

$$\leq \frac{C}{|J|} \int_0^{|J|} \frac{dt}{\sqrt{t}} \int_0^{2|J|} \frac{dx}{\sqrt{x}} \leq C.$$

If  $x_J \geq |J|$ , again by (9) and (10) we can write

$$\begin{aligned} \frac{(x_J + |J|)^2}{|J|^3} \int_0^{|J|} \int_J t \left| \int_0^\infty |D_{\lambda,x} P_t^\lambda(x,y) dy| dx dt &\leq C \frac{(x_J + |J|)^2}{|J|^3} \int_0^{|J|} t dt \int_{x_J - |J|/2}^{x_J + |J|/2} \frac{dx}{x^2} \\ &\leq C \frac{(x_J + |J|)^2}{|J|} \left( \frac{1}{x_J - |J|/2} - \frac{1}{x_J + |J|/2} \right) \leq C \frac{(x_J + |J|)^2}{(x_J + |J|/2)(x_J - |J|/2)} \\ &\leq C \frac{x_J + |J|}{2x_J - |J|} \leq C. \end{aligned}$$

Note that the constant  $C$  does not depend on  $J$ . Thus, (8) is established.  $\square$

By considering Lemma 2.3 and the estimate for  $|f_{2I}|$  we deduce that

$$\frac{1}{|I|} \int_0^{|I|} \int_I |t D_{\lambda,x} P_t^\lambda(f_3)(x)|^2 \frac{dx dt}{t} \leq C \|f\|_{BMO_o(\mathbb{R})}^2.$$

Property (4) is established and we conclude that  $\rho_f$  is a Carleson measure on  $(0, \infty) \times (0, \infty)$  and

$$\|\rho_f\|_{\mathcal{C}} \leq C \|f\|_{BMO_o(\mathbb{R})}^2.$$

Thus the proof of (i)  $\implies$  (ii) in Theorem 1 is finished.

### 3. PROOF OF (ii) $\implies$ (i) IN THEOREM 1

We start this section showing the following characterization of  $BMO_o(\mathbb{R})$  which we need later. Its proof follows the arguments in [6, Theorem 1.1] with minor modifications.

**Lemma 3.1.** *Let  $\lambda > 0$ . Suppose  $f \in L_{loc}^1[0, \infty)$ . Then, the following assertions are equivalent.*

- (i)  $f \in BMO_o(\mathbb{R})$ .
- (ii)  $x^\lambda(1+x^2)^{-\lambda-1}f \in L^1(0, \infty)$  and

$$d\gamma_f(x, t) = |t \partial_t P_t^\lambda(f)(x)|^2 \frac{dx dt}{t}$$

is a Carleson measure on  $(0, \infty) \times (0, \infty)$ .

Moreover, the quantities  $\|f\|_{BMO_o(\mathbb{R})}^2$  and  $\|\gamma_f\|_{\mathcal{C}}$  are equivalent.

*Proof.* (i)  $\implies$  (ii). It follows from Theorem C.

(ii)  $\implies$  (i). We can proceed as in [6, Section 4] by establishing the result in [6, Proposition 4.4] for the new conditions on  $f$ . Actually, we only have to take into account the following estimations.

Let  $a$  be an (odd)-atom, that is, a measurable function satisfying one of the next properties:

(a)  $a = \delta^{-1} \chi_{(0, \delta)}$ , for some  $\delta > 0$ ;

(b) there exists a bounded interval  $I \subset (0, \infty)$  such that  $\text{supp } a \subset I$ ,  $\int_I a(x) dx = 0$  and  $\|a\|_{L^\infty(0, \infty)} \leq |I|^{-1}$ .

We have that

$$(11) \quad \int_0^\infty |t \partial_t P_t^\lambda(y, z) \partial_t P_t^\lambda(a)(y)| dy \leq C \frac{z^\lambda}{(1+z^2)^{\lambda+1}}, \quad z, t \in (0, \infty),$$

with  $C$  independent of  $z$ .

Indeed, since

$$\begin{aligned} \partial_t P_t^\lambda(x, y) &= \frac{2\lambda}{\pi} (xy)^\lambda \left[ \int_0^\pi \frac{(\sin \theta)^{2\lambda-1}}{((x-y)^2 + t^2 + 2xy(1-\cos \theta))^{\lambda+1}} d\theta \right. \\ &\quad \left. - 2(\lambda+1)t^2 \int_0^\pi \frac{(\sin \theta)^{2\lambda-1}}{((x-y)^2 + t^2 + 2xy(1-\cos \theta))^{\lambda+2}} d\theta \right], \quad x, y, t \in (0, \infty), \end{aligned}$$

we get

$$(12) \quad |\partial_t P_t^\lambda(x, y)| \leq C \int_0^\pi \frac{(xy)^\lambda (\sin \theta)^{2\lambda-1}}{((x-y)^2 + t^2 + 2xy(1-\cos \theta))^{\lambda+1}} d\theta \leq C \frac{(xy)^\lambda}{((x-y)^2 + t^2)^{\lambda+1}}, \quad x, y, t \in (0, \infty),$$

and also

$$(13) \quad |\partial_t P_t^\lambda(x, y)| \leq \frac{C}{(x-y)^2 + t^2}, \quad x, y, t \in (0, \infty).$$



Assume that  $\text{supp } a \subset (0, \alpha)$  for certain  $\alpha > 0$ . Then,

$$\begin{aligned} |\partial_t P_t^\lambda(a)(y)| &\leq \|a\|_{L^\infty(0, \infty)} \int_0^\alpha \frac{(yz)^\lambda}{((y-z)^2 + t^2)^{\lambda+1}} dz \\ &\leq Cy^\lambda \begin{cases} t^{-2\lambda-2} \int_0^\alpha z^\lambda dz, & 0 < y \leq 2\alpha, \\ (y^2 + t^2)^{-\lambda-1} \int_0^\alpha z^\lambda dz, & y \geq 2\alpha, \end{cases} \\ &\leq C \frac{y^\lambda}{(1+y^2)^{\lambda+1}}, \quad y, t \in (0, \infty), \end{aligned}$$

where  $C > 0$  does not depend on  $y$ . Hence, by using (12) and (13) it follows that

$$\begin{aligned} &\int_0^\infty |t \partial_t P_t^\lambda(y, z) \partial_t P_t^\lambda(a)(y)| dy \\ &\leq C \left[ \left( \int_0^{z/2} + \int_{2z}^\infty \right) \frac{t(yz)^\lambda}{((y-z)^2 + t^2)^{\lambda+1}} \frac{y^\lambda}{(1+y^2)^{\lambda+1}} dy + \int_{z/2}^{2z} \frac{t}{(y-z)^2 + t^2} \frac{y^\lambda}{(1+y^2)^{\lambda+1}} dy \right] \\ &\leq Ct z^\lambda \left[ \frac{1}{(z^2 + t^2)^{\lambda+1}} \int_0^{z/2} \frac{y^{2\lambda}}{(1+y^2)^{\lambda+1}} dy + \frac{1}{(1+z^2)^{\lambda+1}} \int_{2z}^\infty \frac{y^{2\lambda}}{(y^2 + t^2)^{\lambda+1}} dy \right. \\ &\quad \left. + \frac{1}{(1+z^2)^{\lambda+1}} \int_{z/2}^{2z} \frac{dy}{(x-y)^2 + t^2} dy \right] \leq C \frac{z^\lambda}{(1+z^2)^{\lambda+1}}, \quad z, t \in (0, \infty). \end{aligned}$$

Here, the constant  $C$  can depend on  $t$ , but is independent of  $z$ .

On the other hand we need to estimate  $\sup_{t>0} |M_t^\lambda(a)(z)|$ ,  $z \in (0, \infty)$ , where

$$M_t^\lambda(a) = \frac{1}{4} [t \partial_v P_{2v}^\lambda(a)|_{v=t} - P_{2t}^\lambda(a)], \quad t \in (0, \infty).$$

According to [6, p. 492] we have that

$$\sup_{t>0} |M_t^\lambda(a)(z)| \leq C \begin{cases} 1, & 0 < z \leq 2\alpha, \\ z^{-\lambda-2}, & z \geq 2\alpha. \end{cases} \leq C \begin{cases} 1, & 0 < z \leq 2\alpha, \\ \frac{z^\lambda}{(1+z^2)^{\lambda+1}}, & z \geq 2\alpha, \end{cases}$$

which allows us to obtain

$$(14) \quad \int_0^\infty |f(z)| \sup_{t>0} |M_t^\lambda(a)(z)| dz \leq \int_0^{2\alpha} |f(z)| dz + \int_{2\alpha}^\infty \frac{z^\lambda |f(z)|}{(1+z^2)^{\lambda+1}} dz < \infty.$$

By using (11) and (14) and proceeding as in [6, Section 4] we conclude our result.  $\square$

Assume that  $u$  is a  $\lambda$ -harmonic function on  $(0, \infty) \times (0, \infty)$  such that  $x^{-\lambda}u(x, t) \in C^\infty(\mathbb{R} \times (0, \infty))$  is even in the  $x$ -variable and that the measure

$$d\mu_\lambda(x, t) = |t \nabla_\lambda u(x, t)|^2 \frac{dx dt}{t}$$

is Carleson on  $(0, \infty) \times (0, \infty)$ .

The function  $u$  satisfies the equation

$$\partial_t^2 u + \partial_x^2 u - \frac{\lambda(\lambda-1)}{x^2} u = 0,$$

in a weak sense on  $\mathbb{R} \times (0, \infty)$ , that is, for every  $\phi \in C_c^\infty(\mathbb{R} \times (0, \infty))$ , the space of smooth functions having compact support on  $\mathbb{R} \times (0, \infty)$ ,

$$(15) \quad 0 = \int_{\mathbb{R} \times (0, \infty)} \left( \partial_t u(x, t) \partial_t \phi(x, t) + \partial_x u(x, t) \partial_x \phi(x, t) + \frac{\lambda(\lambda-1)}{x^2} u(x, t) \phi(x, t) \right) dx dt.$$

Indeed, let  $\phi \in C_c^\infty(\mathbb{R} \times (0, \infty))$ . We choose  $0 < a < \infty$  and  $0 < b_1 < b_2 < \infty$  such that  $\text{supp}(\phi) \subset [-a, a] \times [b_1, b_2]$  and define  $v(x, t) = x^{-\lambda}u(x, t)$ ,  $(x, t) \in \mathbb{R} \times (0, \infty)$ .

Since  $v \in C^2(\mathbb{R} \times (0, \infty))$  and  $\lambda > 1$ , we have that  $\frac{u}{x^2}$ ,  $\partial_x u$  and  $\partial_t u$  are in  $L_{\text{loc}}^1(\mathbb{R} \times (0, \infty))$ , and  $\lim_{x \rightarrow 0} \partial_x u(x, t) = 0$ , for every  $t \in (0, \infty)$ . Moreover,

$$\partial_t^2 u(x, t) + \partial_x^2 u(x, t) - \frac{\lambda(\lambda-1)}{x^2} u(x, t) = 0, \quad (x, t) \in (\mathbb{R} \setminus \{0\}) \times (0, \infty).$$

Then, we can write

$$\begin{aligned}
& \int_{\mathbb{R} \times (0, \infty)} \left( \partial_t u(x, t) \partial_t \phi(x, t) + \partial_x u(x, t) \partial_x \phi(x, t) + \frac{\lambda(\lambda - 1)}{x^2} u(x, t) \phi(x, t) \right) dx dt \\
&= \lim_{\varepsilon \rightarrow 0^+} \int_{b_1}^{b_2} \left( \int_{-a}^{-\varepsilon} + \int_{\varepsilon}^a \right) \left( \partial_t u(x, t) \partial_t \phi(x, t) + \partial_x u(x, t) \partial_x \phi(x, t) + \frac{\lambda(\lambda - 1)}{x^2} u(x, t) \phi(x, t) \right) dx dt \\
&= \lim_{\varepsilon \rightarrow 0^+} \int_{b_1}^{b_2} \left( \int_{-a}^{-\varepsilon} + \int_{\varepsilon}^a \right) \left( -\partial_t^2 u(x, t) - \partial_x^2 u(x, t) + \frac{\lambda(\lambda - 1)}{x^2} u(x, t) \right) \phi(x, t) dx dt = 0.
\end{aligned}$$

Since (15) holds, by proceeding as in [23, Lemma 2.6] (see also [49, Lemma 2.1]) we can prove that the function  $u^2$  is subharmonic in  $\mathbb{R} \times (0, \infty)$ . Hence, for every  $x_0 \in \mathbb{R}$ ,  $t_0 \in (0, \infty)$  and  $0 < r < t_0$ ,

$$|u(x_0, t_0)| \leq \left( \frac{1}{\pi r^2} \int_{B((x_0, t_0), r)} |u(x, t)|^2 dx dt \right)^{1/2}.$$

It is clear that  $\partial_t u$  satisfies the same properties than  $u$ . Then, for every  $x_0 \in \mathbb{R}$ ,  $t_0 \in (0, \infty)$  and  $0 < r < t_0$ ,

$$|\partial_t u(x_0, t_0)| \leq \left( \frac{1}{\pi r^2} \int_{B((x_0, t_0), r)} |\partial_t u(x, t)|^2 dx dt \right)^{1/2}.$$

Since the measure  $t|\partial_t u(x, t)|^2 dx dt$  is Carleson on  $(0, \infty) \times (0, \infty)$  we have that, for every  $x_0, t_0 \in (0, \infty)$ ,

$$\begin{aligned}
|(\partial_s u(x_0, s))|_{s=t_0}| &\leq C \left( \frac{1}{t_0^2} \int_{B((x_0, t_0), t_0/2)} |\partial_t u(x, t)|^2 dx dt \right)^{1/2} \leq C \left( \frac{1}{t_0^2} \int_{t_0/2}^{3t_0/2} \int_{x_0-t_0/2}^{x_0+t_0/2} |\partial_t u(x, t)|^2 dx dt \right)^{1/2} \\
(16) \quad &\leq \frac{C}{t_0} \left( \frac{1}{t_0} \int_0^{3t_0/2} \int_{I(x_0, t_0)} t |\partial_t u(x, t)|^2 dx dt \right)^{1/2} \leq \frac{C}{t_0} \|t |\partial_t u(x, t)|^2 dx dt\|_{\mathcal{C}}^{1/2},
\end{aligned}$$

where  $I(x_0, t_0) = (x_0 - \frac{3t_0}{4}, x_0 + \frac{3t_0}{4}) \cap (0, \infty)$ . We have used that  $|\partial_t u(x, t)| = |\partial_t u(-x, t)|$ ,  $x \in \mathbb{R}$  and  $t \in (0, \infty)$ .

From (16) we deduce that, for every  $t_0 > 0$ , there exists  $C > 0$  such that

$$(17) \quad |\partial_t u(x, t)| \leq C, \quad x \in \mathbb{R} \text{ and } t \geq t_0.$$

Our next objective is to show that, for every  $t_0 > 0$ ,

$$(18) \quad \partial_t u(x, t + t_0) = P_t^\lambda((\partial_s u(\cdot, s))|_{s=t_0})(x), \quad x, t \in (0, \infty).$$

In order to see this property we establish previously some results.

**Lemma 3.2.** *Let  $\lambda > 0$ . Suppose that  $f$  is a continuous function on  $(0, \infty)$  such that*

$$\int_0^\infty \frac{y^\lambda |f(y)|}{(1+y^2)^{\lambda+1}} dy < \infty.$$

*Then, the function*

$$v(x, t) = \begin{cases} P_t^\lambda(f)(x), & x, t \in (0, \infty), \\ f(x), & x \in (0, \infty), t = 0 \end{cases}$$

*is  $\lambda$ -harmonic in  $(0, \infty) \times (0, \infty)$  and continuous in  $(0, \infty) \times [0, \infty)$ .*

*Proof.* Differentiating under the integral sign and using [42, (16.1')] it is not hard to see that  $v$  is  $\lambda$ -harmonic function on  $(0, \infty) \times (0, \infty)$ .

Suppose firstly that  $f$  is bounded in  $(0, \infty)$ . Let  $x_0 \in (0, \infty)$ . We write the following decomposition

$$\begin{aligned}
P_t^\lambda(f)(x) - f(x_0) &= \int_0^\infty P_t^\lambda(x, y) [f(y) - f(x_0)] dy + \left( \int_0^\infty P_t^\lambda(x, y) dy - 1 \right) f(x_0) \\
&=: I_1(x, t) + I_2(x, t), \quad x, t \in (0, \infty).
\end{aligned}$$

Assume that  $\varepsilon > 0$ . There exists  $\delta \in (0, x_0/2)$  such that  $|f(y) - f(x_0)| < \varepsilon$  provided that  $|y - x_0| < \delta$ , because  $f$  is continuous in  $x_0$ . Since  $f$  is bounded in  $(0, \infty)$  we get

$$|I_1(x, t)| \leq \left( \int_{|y-x_0|<\delta} + \int_{|y-x_0|\geq\delta} \right) P_t^\lambda(x, y) |f(y) - f(x_0)| dy$$

$$\leq \varepsilon \int_{|y-x_0|<\delta} P_t^\lambda(x, y) dy + 2\|f\|_{L^\infty(0, \infty)} \int_{|y-x_0|\geq\delta} P_t^\lambda(x, y), \quad x, t \in (0, \infty).$$

By [42, p. 86, (b)] we obtain

$$\int_{|y-x_0|<\delta} P_t^\lambda(x, y) dy \leq C \int_{-\infty}^{+\infty} \frac{t}{(x-y)^2 + t^2} dy \leq C, \quad x, t \in (0, \infty),$$

and

$$\begin{aligned} \int_{|y-x_0|\geq\delta} P_t^\lambda(x, y) dy &\leq C \int_{|y-x_0|\geq\delta} \frac{t}{(x-y)^2 + t^2} dy \leq Ct \int_{|y-x_0|\geq\delta} \frac{dy}{(x-y)^2} \\ &\leq Ct \int_{|y-x|\geq\delta/2} \frac{dy}{(x-y)^2} \leq C \frac{t}{\delta}, \quad |x-x_0| < \frac{\delta}{2} \text{ and } t > 0. \end{aligned}$$

Hence,

$$(19) \quad |I_1(x, t)| \leq C \left( \varepsilon + \frac{t}{\delta} \right), \quad |x-x_0| < \frac{\delta}{2} \text{ and } t > 0.$$

On the other hand, by taking into account that  $\int_0^\infty x^{-\lambda} y^\lambda P_t^\lambda(x, y) dy = 1$ ,  $x, t \in (0, \infty)$ , (see, [26, p. 29 (4)], [34, §2 (1), (2)] and [42, (16.1')]), we get

$$\left| \int_0^\infty P_t^\lambda(x, y) dy - 1 \right| \leq \int_0^\infty \left| 1 - \left( \frac{y}{x} \right)^\lambda \right| P_t^\lambda(x, y) dy.$$

We choose  $\eta \in (0, 1)$  such that  $|1 - z^\lambda| < \varepsilon$  provided that  $|1 - z| < \eta$ . From [42, p. 86, (b)] we deduce that

$$\begin{aligned} \left| \int_0^\infty P_t^\lambda(x, y) dy - 1 \right| &\leq \left( \int_0^{(1-\eta)x} + \int_{(1-\eta)x}^{(1+\eta)x} + \int_{(1+\eta)x}^\infty \right) \left| 1 - \left( \frac{y}{x} \right)^\lambda \right| P_t^\lambda(x, y) dy \\ &\leq C \left( \int_0^{(1-\eta)x} \frac{(1 + (1-\eta)^\lambda)t}{(x-y)^2 + t^2} dy + \varepsilon \int_{(1-\eta)x}^{(1+\eta)x} \frac{t}{(x-y)^2 + t^2} dy \right. \\ &\quad \left. + \int_{(1+\eta)x}^\infty \left( \left( \frac{y}{x} \right)^\lambda - 1 \right) \frac{t(xy)^\lambda}{((x-y)^2 + t^2)^{\lambda+1}} dy \right) \\ &\leq C \left( \frac{(1 + (1-\eta)^\lambda)t}{(\eta x)^2} \int_0^{(1-\eta)x} dy + \varepsilon + t \int_{(1+\eta)x}^\infty \frac{(y^\lambda - x^\lambda)y^\lambda}{\left( \frac{\eta y}{1+\eta} \right)^{2\lambda+2}} dy \right) \\ &\leq C \left( \frac{t}{\eta^2 x} + \varepsilon + \frac{t}{\eta^{2\lambda+2}} \int_{(1+\eta)x}^\infty \frac{dy}{y^2} \right) \leq C \left( \varepsilon + \frac{t}{\eta^{2\lambda+2} x} \right) \\ &\leq C \left( \varepsilon + \frac{t}{\eta^{2\lambda+2}} \right), \quad |x-x_0| < \frac{x_0}{2} \text{ and } t > 0. \end{aligned}$$

Then

$$(20) \quad |I_2(x, t)| \leq C \left( \varepsilon + \frac{t}{\eta^{2\lambda+2}} \right), \quad |x-x_0| < \frac{x_0}{2} \text{ and } t > 0.$$

Putting together (19) and (20) we conclude that

$$\lim_{\substack{(x,t) \rightarrow (x_0,0) \\ x,t \in (0,\infty)}} P_t^\lambda(f)(x) = f(x_0).$$

We now study the general case, that is, consider  $f$  a continuous function such that

$$\int_0^\infty \frac{y^\lambda |f(y)|}{(1+y^2)^{\lambda+1}} dy < \infty.$$

Let  $x_0 \in (0, \infty)$ . For every  $n \in \mathbb{N}$  we denote by  $\phi_n$  a smooth function on  $(0, \infty)$  such that  $\phi_n(x) = 1$ ,  $x \in (1/n, n)$ , and  $\phi_n(x) = 0$ ,  $x \in (0, \infty) \setminus (1/(n+1), n+1)$ .

Suppose that  $\varepsilon > 0$  and let  $n_0 \in \mathbb{N}$  such that  $x_0 \in (1/n_0, n_0)$ . We can write

$$(21) \quad \begin{aligned} |P_t^\lambda(f)(x) - f(x_0)| &\leq |P_t^\lambda(f - f\phi_n)(x)| + |P_t^\lambda(f\phi_n)(x) - (f\phi_n)(x_0)| + |(f\phi_n)(x_0) - f(x_0)| \\ &= |P_t^\lambda(f - f\phi_n)(x)| + |P_t^\lambda(f\phi_n)(x) - (f\phi_n)(x_0)|, \quad n \geq n_0. \end{aligned}$$

According to [42, p. 86 (b)] we have that, for each  $|x - x_0| < x_0/2$ ,  $t \in (0, 1)$  and  $n \in \mathbb{N}$ ,  $n \geq 4n_0$ ,

$$\begin{aligned} |P_t^\lambda(f - f\phi_n)(x)| &\leq Ctx^\lambda \left( \int_0^{1/n} + \int_n^\infty \right) \frac{y^\lambda |f(y)|}{((x-y)^2 + t^2)^{\lambda+1}} dy \\ &\leq Cx_0^\lambda \left( \int_0^{1/n} \frac{y^\lambda |f(y)|}{(x_0^2 + t^2)^{\lambda+1}} dy + \int_n^\infty \frac{y^\lambda |f(y)|}{(y^2 + t^2)^{\lambda+1}} dy \right) \\ &\leq C \left( \frac{1}{x_0^{\lambda+2}} \int_0^{1/n} y^\lambda |f(y)| dy + x_0^\lambda \int_n^\infty \frac{y^\lambda |f(y)|}{y^{2\lambda+2}} dy \right) \\ &\leq C \left( \frac{1}{x_0^{\lambda+2}} \int_0^{1/n} \frac{y^\lambda |f(y)|}{(1+y^2)^{\lambda+1}} dy + x_0^\lambda \int_n^\infty \frac{y^\lambda |f(y)|}{(1+y^2)^{\lambda+1}} dy \right), \end{aligned}$$

with  $C$  independent of  $x$ ,  $t$  and  $n$ .

Then, we can find  $n_1 \in \mathbb{N}$ ,  $n_1 \geq 4n_0$ , such that

$$(22) \quad |P_t^\lambda(f - f\phi_n)(x)| < \varepsilon, \quad |x - x_0| < \frac{x_0}{2}, \quad t \in (0, 1), \quad n \in \mathbb{N}, \quad n \geq n_1.$$

On the other hand, for each  $n \in \mathbb{N}$ , since  $f\phi_n$  is continuous and bounded on  $(0, \infty)$ ,

$$(23) \quad \lim_{\substack{(x,t) \rightarrow (x_0,0) \\ x,t \in (0,\infty)}} P_t^\lambda(f\phi_n)(x) = (f\phi_n)(x_0).$$

By considering (21), (22) and (23) we conclude that

$$\lim_{\substack{(x,t) \rightarrow (x_0,0) \\ x,t \in (0,\infty)}} P_t^\lambda(f)(x) = f(x_0).$$

□

The space of  $\lambda$ -harmonic functions on  $(0, \infty) \times \mathbb{R}$  form a BreLOT harmonic space. Then, it is well-known that  $\lambda$ -harmonic functions on  $(0, \infty) \times \mathbb{R}$  satisfy the mean value properties with respect to the  $\lambda$ -harmonic measures. Recently, Eriksson and Orelma ([27]) have established explicit mean value properties for solutions of Weinstein operators. We recall some results in [27] specified for our particular case and that will be useful.

We consider on  $(0, \infty) \times \mathbb{R}$  the hyperbolic metric  $d_h$  defined by

$$d_h(a, b) = \operatorname{arcosh} \sigma(a, b), \quad a, b \in (0, \infty) \times \mathbb{R},$$

where

$$\sigma(a, b) = \frac{(a_1 - b_1)^2 + (a_2 - b_2)^2 + 2a_1b_1}{2a_1b_1}, \quad a = (a_1, b_1), b = (b_1, b_2) \in (0, \infty) \times \mathbb{R}.$$

The hyperbolic ball  $B_h(a, r)$  with center  $a \in (0, \infty) \times \mathbb{R}$  and radius  $r > 0$  is defined as usual by

$$B_h(a, r) = \{b \in (0, \infty) \times \mathbb{R} : d_h(a, b) < r\}.$$

For every  $a \in (0, \infty) \times \mathbb{R}$  and  $r > 0$ ,  $B_h(a, r)$  is actually an Euclidean ball. We have that, for each  $a = (a_1, a_2) \in (0, \infty) \times \mathbb{R}$  and  $r > 0$

$$B_h(a, r) = \{b \in (0, \infty) \times \mathbb{R} : |\tilde{a} - b| < a_1 \sinh r\},$$

where  $\tilde{a} = (a_1 \cosh r, a_2)$ .

In [1] Akin and Leutwiler introduced the function

$$\varphi_\alpha(r) = \frac{(1 - r^2)^\alpha}{2} \int_{-1}^1 \frac{dy}{|r - y|^{2\alpha}}, \quad 0 < r < 1,$$

in their investigations about Weinstein equations.

From [27, Theorem 3.3] it follows the following mean value property for  $\lambda$ -harmonic functions.

**Lemma 3.3.** *Let  $\lambda > 0$ . Assume that  $U$  is an open subset of  $(0, \infty) \times \mathbb{R}$ . If  $v$  is a  $\lambda$ -harmonic function in  $U$  then, for every  $a \in U$  and  $r > 0$  such that  $B_h(a, r) \subset U$ ,*

$$(24) \quad v(a) = \frac{1}{2 \sinh(r) \varphi_\alpha(\tanh(r/2))} \int_{\partial B_h(a, r)} v(b_1, b_2) \frac{d\tau(b_1, b_2)}{b_1},$$

where  $\alpha = (1 + |2\lambda - 1|)/2$  and  $\tau$  denotes the length measure on  $\partial B_h(a, r)$ .

We now prove the converse of Lemma 3.3.

**Lemma 3.4.** *Let  $\lambda > 0$  and let  $U$  be an open subset of  $(0, \infty) \times \mathbb{R}$ . Suppose that  $v$  is a continuous function on  $U$  such that the mean value property (24) holds for every  $a \in U$  and  $r > 0$  such that  $\overline{B_h(a, r)} \subset U$ . Then,  $v$  is  $\lambda$ -harmonic in  $U$ .*

*Proof.* In order to show this property we follow a procedure similar to the classical one used to establish the corresponding result for harmonic functions.

In a first step we prove a maximum principle in this context. Let  $a \in U$  and  $r > 0$  such that  $\overline{B_h(a, r)} \subset U$ . Since  $v$  is continuous in  $\overline{B_h(a, r)}$ , the set

$$A = \{b \in \overline{B_h(a, r)} : v(b) \geq v(c), c \in B_h(a, r)\} \neq \emptyset.$$

Suppose that  $A \cap \partial B_h(a, r) = \emptyset$ . Since  $A$  is closed,

$$d(A, \partial B_h(a, r)) = \min\{|c - z| : c \in A, z \in \partial B_h(a, r)\} > 0.$$

We choose  $b \in A$  such that

$$d(b, \partial B_h(a, r)) = \inf\{|b - z| : z \in \partial B_h(a, r)\} = d(A, \partial B_h(a, r))$$

and  $R > 0$  such that  $B_h(b, R) \subset B_h(a, r)$ . We consider the sets

$$M_+ = A \cap \partial B_h(b, R) \quad \text{and} \quad M_- = A^c \cap \partial B_h(b, R).$$

Since  $\tau(M_-) > 0$  we deduce that

$$\begin{aligned} \frac{1}{2 \sinh(R) \varphi_\alpha(\tanh(R/2))} \int_{\partial B_h(b, R)} v(z_1, z_2) \frac{d\tau(z_1, z_2)}{z_1} \\ = \frac{1}{2 \sinh(R) \varphi_\alpha(\tanh(R/2))} \left( \int_{M_+} + \int_{M_-} \right) v(z_1, z_2) \frac{d\tau(z_1, z_2)}{z_1} < v(b). \end{aligned}$$

We have taken into account that

$$\int_{\partial B_h(b, R)} \frac{d\tau(z_1, z_2)}{z_1} = 2 \sinh(R) \varphi_\alpha\left(\tanh \frac{R}{2}\right).$$

Hence, since  $v$  satisfies (24) for every  $a \in U$  and  $r > 0$  such that  $\overline{B_h(a, r)} \subset U$ ,  $A \cap \partial B_h(a, r) \neq \emptyset$ . Then,

$$\max_{b \in \overline{B_h(a, r)}} v(b) = \max_{b \in \partial B_h(a, r)} v(b).$$

We now observe that the operator

$$\mathcal{L}_\lambda = \partial_t^2 + \partial_x^2 - \frac{\lambda(\lambda - 1)}{x^2},$$

is uniformly elliptic on every bounded domain  $\Omega$  such that  $\overline{\Omega} \subset (0, \infty) \times \mathbb{R}$ . Then, for every  $b \in U$  and  $R > 0$  such that  $\overline{B_h(b, R)} \subset U$  and every continuous function  $f$  on  $\partial B_h(b, R)$ , there exists a continuous function  $w$  in  $\overline{B_h(b, R)}$  such that  $w|_{\partial B_h(b, R)} = f$  and  $w$  is  $\lambda$ -harmonic in  $B_h(b, R)$ . Hence, according to Lemma 3.3, this function  $w$  satisfies the mean value property (24) for every  $a \in B_h(b, r)$  and  $r > 0$  such that  $\overline{B_h(a, r)} \subset B_h(b, R)$ .

Let  $b \in U$  and  $R > 0$  such that  $\overline{B_h(b, R)} \subset U$ . We define  $f = v|_{\partial B_h(b, R)}$  and denote by  $w$  the continuous function in  $\overline{B_h(b, R)}$  such that  $w|_{\partial B_h(b, R)} = f$  and  $w$  is  $\lambda$ -harmonic in  $B_h(b, R)$ . We consider the function  $F = v - w$  in  $\overline{B_h(b, R)}$ . It is clear that  $F|_{\partial B_h(b, R)} = 0$  and  $F$  satisfies the mean value property (24) for every  $a \in B_h(b, R)$  and  $r > 0$  such that  $\overline{B_h(a, r)} \subset B_h(b, R)$ . The maximum (minimum) property allows us to conclude that  $v = w$  in  $\overline{B_h(b, R)}$ . Thus, we prove that  $v$  is  $\lambda$ -harmonic in  $U$ .  $\square$

**REMARK** As it can be deduced from the proof of Lemma 3.4, in order to see that a function  $v$  continuous in an open subset  $U$  of  $(0, \infty) \times \mathbb{R}$  is  $\lambda$ -harmonic in  $U$ , it is sufficient to show that, for every  $a \in U$ , there exists a sequence  $(r_n)_{n \in \mathbb{N}} \subset (0, \infty)$  such that,  $r_n \rightarrow 0$ , as  $n \rightarrow \infty$ , that  $\overline{B_h(a, r_n)} \subset U$ ,  $n \in \mathbb{N}$ , and

$$v(a) = \frac{1}{2 \sinh(r_n) \varphi_\alpha(\tanh(r_n/2))} \int_{\partial B_h(a, r_n)} v(b_1, b_2) \frac{d\tau(b_1, b_2)}{b_1},$$

with  $\alpha = (1 + |\lambda - 1|)/2$ .

Now we establish a uniqueness result for  $\lambda$ -harmonic functions in  $(0, \infty) \times (0, \infty)$ .

**Lemma 3.5.** *Let  $\lambda > 1$ . Suppose that  $v$  is a bounded and continuous function on  $(0, \infty) \times [0, \infty)$  such that  $v$  is  $\lambda$ -harmonic in  $(0, \infty) \times (0, \infty)$  and  $v(x, 0) = 0$ ,  $x \in (0, \infty)$ . Then,  $v = 0$  in  $(0, \infty) \times [0, \infty)$ .*

*Proof.* We define

$$w(x, t) = \begin{cases} v(x, t), & x \in (0, \infty), t \in [0, \infty) \\ -v(x, -t), & x \in (0, \infty), t \in (-\infty, 0). \end{cases}$$

$w$  is a continuous function in  $(0, \infty) \times \mathbb{R}$ . Moreover,  $w$  is  $\lambda$ -harmonic in  $(0, \infty) \times \mathbb{R} \setminus \{0\}$ . According to Lemma 3.4, in order to see that  $w$  is  $\lambda$ -harmonic in  $(0, \infty) \times \mathbb{R}$  it is sufficient to observe that, for every  $x \in (0, \infty)$  and  $r > 0$  such that  $\overline{B_h((x, 0), r)} \subset (0, \infty) \times \mathbb{R}$ ,

$$0 = \int_{\partial B_h((x, 0), r)} w(b_1, b_2) \frac{d\tau(b_1, b_2)}{b_1}.$$

Note that this property holds because  $w$  is odd in the second variable and every hyperbolic ball centered in the line  $(0, \infty) \times \{0\}$  is actually an Euclidean ball with center in the same line.

Since  $v$  is bounded in  $(0, \infty) \times [0, \infty)$ ,  $w$  is also bounded in  $(0, \infty) \times \mathbb{R}$ . Then, there exists  $M > 0$  such that  $|w(x, t)| \leq M$ ,  $x \in (0, \infty)$  and  $t \in \mathbb{R}$ . The function  $g(x, t) = x^\lambda + x^{1-\lambda}$ ,  $x \in (0, \infty)$  and  $t \in \mathbb{R}$ , is  $\lambda$ -harmonic in  $(0, \infty) \times \mathbb{R}$ . We define the function

$$\tilde{w}(x, t) = w(x, t) + M(x^\lambda + x^{1-\lambda}), \quad x \in (0, \infty) \text{ and } t \in \mathbb{R}.$$

Thus,  $\tilde{w}(x, t) \geq 0$ ,  $x \in (0, \infty)$  and  $t \in \mathbb{R}$ , and  $\tilde{w}$  is  $\lambda$ -harmonic in  $(0, \infty) \times \mathbb{R}$ . According to [39, Theorem 2.2] there exists a positive  $\sigma$ -finite measure  $\gamma$  on  $\mathbb{R}$  and  $m \geq 0$  such that

$$\tilde{w}(x, t) = x^\lambda \left( m + \int_{-\infty}^{+\infty} \frac{d\gamma(s)}{((t-s)^2 + x^2)^\lambda} \right), \quad x \in (0, \infty) \text{ and } t \in \mathbb{R}.$$

Then,

$$w(x, t) = -M(x^\lambda + x^{1-\lambda}) + x^\lambda \left( m + \int_{-\infty}^{+\infty} \frac{d\gamma(s)}{((t-s)^2 + x^2)^\lambda} \right), \quad x \in (0, \infty) \text{ and } t \in \mathbb{R}.$$

Since  $w(x, 0) = 0$ ,  $x \in (0, \infty)$ , we have that

$$-Mx^{1-2\lambda} + m - M + \int_{-\infty}^{+\infty} \frac{d\gamma(s)}{(s^2 + x^2)^\lambda} = 0, \quad x \in (0, \infty).$$

By letting  $x \rightarrow +\infty$  and by dominated convergence theorem we deduce that  $m = M$ . Hence,

$$(25) \quad w(x, t) = x^{1-\lambda} \left( -M + \int_{-\infty}^{+\infty} \frac{x^{2\lambda-1}}{((t-s)^2 + x^2)^\lambda} d\gamma(s) \right), \quad x \in (0, \infty) \text{ and } t \in \mathbb{R},$$

and again, since  $w(x, 0) = 0$ ,  $x \in (0, \infty)$ , we deduce that

$$(26) \quad M = \int_{-\infty}^{+\infty} \frac{x^{2\lambda-1}}{(s^2 + x^2)^\lambda} d\gamma(s), \quad x \in (0, \infty).$$

By using Radon-Nikodym theorem we can write  $d\gamma(s) = hds + d\mu(s)$ , where  $0 \leq h \in L^1_{\text{loc}}(\mathbb{R})$  and  $\mu$  is a positive measure that is orthogonal to the Lebesgue measure on  $\mathbb{R}$ .

It can be seen that

$$(27) \quad \lim_{x \rightarrow 0^+} \int_{-\infty}^{+\infty} \frac{x^{2\lambda-1}}{((t-s)^2 + x^2)^\lambda} d\gamma(s) = Ah(t), \quad \text{a.e. } t \in \mathbb{R}.$$

Here, a.e. is understood with respect to the Lebesgue measure on  $\mathbb{R}$  and

$$A = \int_{-\infty}^{+\infty} \frac{1}{(s^2 + 1)^\lambda} ds = \frac{\sqrt{\pi}\Gamma(\lambda - 1/2)}{\Gamma(\lambda)}.$$

Indeed, fix  $N \in \mathbb{N}$ . It is sufficient to see (27) for a.e.  $|t| \leq N$ . Denote by  $K_x$ ,  $x \in (0, \infty)$ , the kernel

$$K_x(t, s) = \frac{x^{2\lambda-1}}{((t-s)^2 + x^2)^\lambda}, \quad t, s \in \mathbb{R}.$$

For every  $n \in \mathbb{N}$ , let us define  $h_n(t) = h(t)\chi_{(-n, n)}(t)$ ,  $t \in \mathbb{R}$ . Then, since  $\int_{-\infty}^{+\infty} K_x(t, s)ds = A$ ,  $x \in (0, \infty)$ ,  $t \in \mathbb{R}$ , it follows that, for each  $n \in \mathbb{N}$ ,  $n \geq N$ , we can write

$$(28) \quad \begin{aligned} \int_{-\infty}^{+\infty} K_x(t, s)d\gamma(s) - Ah(t) &= \int_{-\infty}^{+\infty} K_x(t, s)[h(s) - h_n(s)]ds + \int_{-\infty}^{+\infty} K_x(t, s)h_n(s)ds - Ah_n(t) \\ &+ \int_{-\infty}^{+\infty} K_x(t, s)d\mu(s), \quad x \in (0, \infty), |t| \leq N. \end{aligned}$$

When  $n \geq 2N$ , the first term can be bounded as follows,

$$\begin{aligned} \left| \int_{-\infty}^{+\infty} K_x(t, s)[h(s) - h_n(s)]ds \right| &\leq \int_{|s|>n} \frac{x^{2\lambda-1}|h(s)|}{((t-s)^2 + x^2)^\lambda} ds \leq C \int_{|s|>n} \frac{|h(s)|}{(s^2 + x^2)^\lambda} ds \\ &\leq C \int_{|s|>n} \frac{|h(s)|}{s^{2\lambda}} ds \leq C \int_{|s|>n} \frac{|h(s)|}{(s^2 + 1)^\lambda} ds, \quad x \in (0, 1), |t| \leq N. \end{aligned}$$

Thus, for every  $\varepsilon > 0$ , there exists  $n_0 \in \mathbb{N}$ ,  $n_0 \geq 2N$ , independent of  $x \in (0, 1)$  and  $|t| \leq N$ , such that

$$(29) \quad \left| \int_{-\infty}^{+\infty} K_x(t, s)[h(s) - h_{n_0}(s)]ds \right| < \varepsilon, \quad x \in (0, 1), |t| \leq N.$$

On the other hand, we observe that

$$|K_x(t, s)| \leq C \begin{cases} \frac{1}{x}, & |t - s| < x, \\ \frac{1}{2^{(2\lambda-1)k} 2^k x}, & 2^{k-1}x \leq |t - s| < 2^k x, \end{cases}, \quad x \in (0, \infty), t, s \in \mathbb{R}, k \in \mathbb{N}.$$

Then, since  $\lambda > 1$ , it is not difficult to see that

$$\sup_{x \in (0, \infty)} \left| \int_{-\infty}^{+\infty} K_x(t, s)h_{n_0}(s)ds \right| \leq C \mathcal{M}(|h_{n_0}|)(t), \quad t \in \mathbb{R},$$

and

$$\sup_{x \in (0, \infty)} \left| \int_{-\infty}^{+\infty} K_x(t, s)d\mu(s) \right| \leq C \mathcal{M}(\mu)(t), \quad t \in \mathbb{R},$$

where  $\mathcal{M}$  represents the classical Hardy-Littlewood maximal function defined on  $L^1(\mathbb{R})$  and on the set of the Borel measures on  $\mathbb{R}$ .

By following standard arguments (see [2, Theorems 6.39 and 6.42], for instance) we obtain that

$$(30) \quad \lim_{x \rightarrow 0^+} \int_{-\infty}^{+\infty} K_x(t, s)h_{n_0}(s)ds = Ah_{n_0}(t), \quad \text{a.e. } t \in \mathbb{R},$$

and

$$(31) \quad \lim_{x \rightarrow 0^+} \int_{-\infty}^{+\infty} K_x(t, s)d\mu(s) = 0, \quad \text{a.e. } t \in \mathbb{R}.$$

Putting together (28), (29), (30) and (31) we obtain (27) for a.e.  $|t| \leq N$ .

By taking into account that  $w$  is a bounded function in  $(0, \infty) \times \mathbb{R}$  and  $\lambda > 1$ , from (25) we deduce that

$$-M + Ah(t) = 0, \quad \text{a.e. } t \in \mathbb{R},$$

and by (26), it follows that

$$\int_{-\infty}^{+\infty} \frac{d\mu(s)}{(s^2 + x^2)^\lambda} = 0, \quad x \in (0, \infty).$$

Hence,  $\mu = 0$ . By using again (25) we obtain

$$w(x, t) = x^{1-\lambda} \left( -M + \frac{M}{A} \int_{-\infty}^{+\infty} \frac{x^{2\lambda-1}}{((t-s)^2 + x^2)^\lambda} ds \right) = 0, \quad x \in (0, \infty) \text{ and } t \in \mathbb{R}.$$

Then  $v(x, t) = 0$ ,  $x \in (0, \infty)$  and  $t \geq 0$ . □

*Proof of (18).* Let  $t_0 > 0$ . We define the function  $v(x, t) = \partial_t u(x, t + t_0)$ ,  $x \in (0, \infty)$  and  $t \in [0, \infty)$ . We have that  $v$  is bounded (see (17)), continuous in  $(0, \infty) \times [0, \infty)$  and  $\lambda$ -harmonic in  $(0, \infty) \times (0, \infty)$ . We consider  $f(x) = v(x, 0)$ ,  $x \in (0, \infty)$ , and define

$$V(x, t) = \begin{cases} P_t^\lambda(f)(x), & x, t \in (0, \infty), \\ f(x), & x \in (0, \infty) \text{ and } t = 0. \end{cases}$$

Since  $f$  is bounded and continuous in  $(0, \infty)$ , by Lemma 3.2, the function  $V$  is continuous and bounded in  $(0, \infty) \times [0, \infty)$  and  $\lambda$ -harmonic in  $(0, \infty) \times (0, \infty)$ . The function  $V - v$  is bounded and continuous in  $(0, \infty) \times [0, \infty)$ , and  $\lambda$ -harmonic in  $(0, \infty) \times (0, \infty)$ . Moreover,  $V(x, 0) = v(x, 0)$ ,  $x \in (0, \infty)$ . According to Lemma 3.5,  $V(x, t) = v(x, t)$ ,  $x \in (0, \infty)$  and  $t \in [0, \infty)$ . Thus, (18) is established. □

Our next objective is to establish that

$$(32) \quad u(x, t+r) = P_t^\lambda(u(\cdot, r))(x), \quad x, t, r \in (0, \infty).$$

We have that, for every  $r > 0$ ,

$$(33) \quad \int_0^\infty \frac{y^\lambda |u(y, r)|}{(1+y^2)^{\lambda+1}} dy < \infty,$$

and then the integral defining  $P_t^\lambda(u(\cdot, r))(x)$  is absolutely convergent, for every  $x, t \in (0, \infty)$ .

In order to show (32) we see previously that

$$(34) \quad \lim_{r \rightarrow \infty} \partial_t \int_0^\infty P_t^\lambda(x, y) u(y, r) dy = 0, \quad x, t \in (0, \infty).$$

We note that the arguments that we will use to prove (34) also allow us to obtain (33).

*Proof of (34).* Since, for every  $x, t \in (0, \infty)$ ,  $\int_0^\infty P_t^\lambda(x, y) y^\lambda dy = x^\lambda$  ([42, p. 84]) we can write

$$(35) \quad \begin{aligned} \partial_t \int_0^\infty P_t^\lambda(x, y) u(y, r) dy &= \partial_t \int_0^\infty P_t^\lambda(x, y) y^\lambda y^{-\lambda} u(y, r) dy \\ &= \partial_t \int_0^\infty P_t^\lambda(x, y) y^\lambda [y^{-\lambda} u(y, r) - x^{-\lambda} u(x, r)] dy \\ &= \partial_t \int_0^\infty P_t^\lambda(x, y) y^\lambda \int_x^y \partial_z [z^{-\lambda} u(z, r)] dz dy, \quad x, t, r \in (0, \infty). \end{aligned}$$

Moreover, we have that

$$\left| \int_x^y \partial_z [z^{-\lambda} u(z, r)] dz \right| \leq \int_x^y |D_{\lambda, z} u(z, r)| z^{-\lambda} dz \leq C |y^{1-\lambda} - x^{1-\lambda}| \sup_{z \in I_{x, y}} |D_{\lambda, z} u(z, r)|, \quad x, y, r \in (0, \infty).$$

Here,  $I_{x, y} = [\min\{x, y\}, \max\{x, y\}]$ ,  $x, y \in (0, \infty)$ .

Since  $u$  is  $\lambda$ -harmonic in  $(0, \infty) \times (0, \infty)$ , we get

$$(\partial_t^2 - D_{\lambda, x} D_{\lambda, x}^*) D_{\lambda, x} u(x, t) = D_{\lambda, x} (\partial_t^2 - D_{\lambda, x}^* D_{\lambda, x}) u(x, t) = 0, \quad x, t \in (0, \infty).$$

Note that

$$-D_\lambda D_\lambda^* = x^\lambda D x^{-2\lambda} D x^\lambda = u'' - \frac{(\lambda+1)\lambda}{x^2} u = B_{\lambda+1}.$$

Then,  $D_{\lambda, x} u$  is  $(\lambda+1)$ -harmonic in  $(0, \infty) \times (0, \infty)$ . Moreover,  $x^{-\lambda-1} D_{\lambda, x} u = \frac{1}{x} \partial_x (x^{-\lambda} u)$  is regular in  $\mathbb{R} \times (0, \infty)$  and even in the  $x$ -variable. By proceeding as in the beginning of Section 3 after Lemma 3.1 we can see that  $(D_{\lambda, x} u)^2$  is subharmonic in  $\mathbb{R} \times (0, \infty)$ .

Let  $x, t \in (0, \infty)$ . The subharmonicity of  $(D_{\lambda, x} u)^2$  allows us to write

$$(36) \quad \begin{aligned} \sup_{z \in I_{x, y}} |D_{\lambda, z} u(z, r)| &\leq C \sup_{z \in I_{x, y}} \left( \frac{1}{r^2} \int_{B((z, r), r/4)} |D_{\lambda, a} u(a, b)|^2 da db \right)^{1/2} \\ &\leq C \left( \frac{1}{r^2} \int_{\frac{3r}{4}}^{\frac{5r}{4}} \int_{x-\frac{5r}{4}}^{x+\frac{5r}{4}} |D_{\lambda, a} u(a, b)|^2 da db \right)^{1/2} \\ &\leq \frac{C}{r} \|b\| D_{\lambda, a} u(a, b)^2 da db \|_{\mathcal{C}}^{1/2}, \quad |x-y| \leq r, \quad y, r \in (0, \infty). \end{aligned}$$

Also, we have that (see [44, Lemma 3.2])

$$(37) \quad |y^{1-\lambda} - x^{1-\lambda}| \leq C |x-y| \frac{\min\{x, y\}^{2-\lambda}}{xy}, \quad y \in (0, \infty).$$

Then, by using (12) we obtain

$$\begin{aligned} &\left| \int_{0, |x-y| \leq r}^\infty \partial_t P_t^\lambda(x, y) y^\lambda \int_x^y \partial_z [z^{-\lambda} u(z, r)] dz dy \right| \\ &\leq C \frac{\|b\| D_{\lambda, a} u(a, b)^2 da db \|_{\mathcal{C}}^{1/2}}{r} \int_{0, |x-y| \leq r}^\infty \frac{x^{\lambda-1} y^{2\lambda-1} |x-y| \min\{x, y\}^{2-\lambda}}{((x-y)^2 + t^2)^{\lambda+1}} dy \\ &\leq \frac{C}{r} \left[ \int_{\max\{0, x-r\}}^x \frac{x^{\lambda-1} y^{\lambda+1} |x-y|}{((x-y)^2 + t^2)^{\lambda+1}} dy + \int_x^{x+r} \frac{xy^{2\lambda-1} |x-y|}{((x-y)^2 + t^2)^{\lambda+1}} dy \right] \end{aligned}$$



$$\begin{aligned}
&\leq \frac{C}{r} \left[ \frac{x^{\lambda-1}}{t^{2\lambda+1}} \int_0^x y^{\lambda+1} dy + \frac{x}{t^{2\lambda+1}} \int_x^{2x} y^{2\lambda-1} dy + x \int_{2x}^\infty \frac{y^{2\lambda}}{(y+t)^{2\lambda+2}} dy \right] \\
(38) \quad &\leq \frac{C}{r} \left[ \left( \frac{x}{t} \right)^{2\lambda+1} + \frac{x}{x+t} \right] \leq \frac{C}{r}, \quad r \in (0, \infty),
\end{aligned}$$

being  $C$  depending on  $x$  and  $t$  but not on  $r$ .

We now make the following decomposition

$$\begin{aligned}
y^{-\lambda}u(y, r) - x^{-\lambda}u(x, r) &= y^{-\lambda}[u(y, r) - u(y, |x-y|)] + y^{-\lambda}u(y, |x-y|) \\
&\quad - x^{-\lambda}u(x, |x-y|) + x^{-\lambda}[u(x, |x-y|) - u(x, r)] \\
&= -y^{-\lambda} \int_r^{|x-y|} \partial_s u(y, s) ds + x^{-\lambda} \int_r^{|x-y|} \partial_s u(x, s) ds \\
&\quad + \int_x^y z^{-\lambda} D_{\lambda, z} u(z, |x-y|) dz, \quad y, r \in (0, \infty).
\end{aligned}$$

We get

$$\begin{aligned}
|y^{-\lambda}u(y, r) - x^{-\lambda}u(x, r)| &\leq C \left[ y^{-\lambda} \int_r^{|x-y|} |\partial_s u(y, s)| ds + x^{-\lambda} \int_r^{|x-y|} |\partial_s u(x, s)| ds \right. \\
&\quad \left. + |y^{1-\lambda} - x^{1-\lambda}| \sup_{z \in I_{x, y}} |D_{\lambda, z} u(z, |x-y|)| \right], \quad |x-y| > r, \quad y, r \in (0, \infty).
\end{aligned}$$

From (37), as in (36), it follows that

$$\begin{aligned}
|y^{1-\lambda} - x^{1-\lambda}| \sup_{z \in I_{x, y}} |D_{\lambda, z} u(z, |x-y|)| &\leq C|x-y| \frac{\min\{x, y\}^{2-\lambda}}{xy} \sup_{z \in I_{x, y}} |D_{\lambda, z} u(z, |x-y|)| \\
&\leq C \frac{\min\{x, y\}^{2-\lambda}}{xy} \|b|D_{\lambda, a} u(a, b)|^2 dadb\|_{\mathcal{C}}^{1/2}, \quad y \in (0, \infty).
\end{aligned}$$

By (16) we get

$$\begin{aligned}
y^{-\lambda} \int_r^{|x-y|} |\partial_s u(y, s)| ds + x^{-\lambda} \int_r^{|x-y|} |\partial_s u(x, s)| ds &\leq C(x^{-\lambda} + y^{-\lambda}) \|b|\partial_b u(a, b)|^2 dadb\|_{\mathcal{C}}^{1/2} \int_r^{|x-y|} \frac{ds}{s} \\
&\leq C \min\{x, y\}^{-\lambda} \log \frac{|x-y|}{r} \|b|\partial_b u(a, b)|^2 dadb\|_{\mathcal{C}}^{1/2}, \quad |x-y| > r, \quad y, r \in (0, \infty).
\end{aligned}$$

Then, by (12),

$$\begin{aligned}
&\left| \int_{0, |x-y| > r}^\infty \partial_t P_t^\lambda(x, y) y^\lambda [y^{-\lambda}u(y, r) - x^{-\lambda}u(x, r)] dy \right| \\
&\leq Cx^\lambda \int_{0, |x-y| > r}^\infty \frac{y^{2\lambda}}{((x-y)^2 + t^2)^{\lambda+1}} \left( \min\{x, y\}^{-\lambda} \log \frac{|x-y|}{r} + \frac{\min\{x, y\}^{2-\lambda}}{xy} \right) dy, \quad r \in (0, \infty).
\end{aligned}$$

We analyze each term separately. We have that, for every  $r > x$ ,

$$x^\lambda \int_{0, |x-y| > r}^\infty \frac{y^{2\lambda} \min\{x, y\}^{2-\lambda}}{((x-y)^2 + t^2)^{\lambda+1}} \frac{dy}{xy} \leq \frac{x}{r} \int_{x+r}^\infty \frac{y^{2\lambda-1}}{(y-x+t)^{2\lambda+1}} dy \leq C \frac{x}{r} \int_{2x}^\infty \frac{dy}{(y-x)^2} \leq \frac{C}{r},$$

and

$$\begin{aligned}
x^\lambda \int_{0, |x-y| > r}^\infty \frac{y^{2\lambda} \min\{x, y\}^{-\lambda}}{((x-y)^2 + t^2)^{\lambda+1}} \log \frac{|x-y|}{r} dy &\leq \int_{x+r}^\infty \frac{y^{2\lambda}}{(y-x+t)^{2\lambda+2}} \log \frac{y-x}{r} dy \\
&\leq C \int_{x+r}^\infty \frac{1}{(y-x)^2} \log \frac{y-x}{r} dy \leq \frac{C}{r} \int_1^\infty \frac{\log u}{u^2} du \leq \frac{C}{r}.
\end{aligned}$$

Here the constant  $C$  can depend on  $x$  and  $t$ , but not on  $r$ .

We conclude that

$$(39) \quad \left| \int_{0, |x-y| > r}^\infty \partial_t P_t^\lambda(x, y) y^\lambda [y^{-\lambda}u(y, r) - x^{-\lambda}u(x, r)] dy \right| \leq \frac{C}{r}, \quad r > x.$$

By combining (35), (38) and (39) we deduce that (34) holds.  $\square$

*Proof of (32).* According to (18) we have that

$$\partial_t u(x, t+r) = \partial_r u(x, t+r) = P_t^\lambda \left[ \partial_r u(\cdot, r) \right](x), \quad x, t, r \in (0, \infty).$$

Since the differentiation under the integral sign is justified by the properties of the function  $u$ , we obtain

$$\partial_r [u(x, t+r) - P_t^\lambda(u(\cdot, r))(x)] = 0, \quad x, t, r \in (0, \infty),$$

and

$$\partial_r [\partial_t u(x, t+r) - \partial_t P_t^\lambda(u(\cdot, r))(x)] = 0, \quad x, t, r \in (0, \infty).$$

From (16) and (34) it follows that

$$\lim_{r \rightarrow \infty} [\partial_t u(x, t+r) - \partial_t P_t^\lambda(u(\cdot, r))(x)] = 0, \quad x, t \in (0, \infty).$$

Then,

$$\partial_t [u(x, t+r) - P_t^\lambda(u(\cdot, r))(x)] = 0, \quad x, t, r \in (0, \infty).$$

Also, (33) and Lemma 3.2 lead to

$$\lim_{t \rightarrow 0^+} (u(x, t+r) - P_t^\lambda(u(\cdot, r))(x)) = 0, \quad x, r \in (0, \infty).$$

We conclude that

$$u(x, t+r) = P_t^\lambda(u(\cdot, r))(x), \quad x, t, r \in (0, \infty),$$

and (32) is proved.  $\square$

For every  $k \in \mathbb{N}$ , we define

$$u_k(x, t) = u\left(x, t + \frac{1}{k}\right), \quad x \in (0, \infty), \quad t \in [0, \infty).$$

We now establish that there exists  $C > 0$  such that

$$(40) \quad \sup_I \frac{1}{|I|} \int_0^{|I|} \int_I t |\partial_t u_k(x, t)|^2 dx dt \leq C \|t |\partial_t u(x, t)|^2 dx dt\|_{\mathcal{C}},$$

where the supremum is taken over all bounded intervals  $I \subset (0, \infty)$ .

*Proof of (40).* Let  $k \in \mathbb{N}$  and let  $I$  be a bounded interval in  $(0, \infty)$ . Suppose that  $|I| \geq 1/k$ . We obtain

$$(41) \quad \begin{aligned} \frac{1}{|I|} \int_0^{|I|} \int_I t |\partial_t u_k(x, t)|^2 dx dt &\leq \frac{1}{|I|} \int_0^{|I|} \int_I \left(t + \frac{1}{k}\right) \left|(\partial_t u)\left(x, t + \frac{1}{k}\right)\right|^2 dx dt \\ &\leq \frac{1}{|I|} \int_0^{2|I|} \int_{\widehat{I}} s |\partial_s u(x, s)|^2 dx ds \leq 2 \|s |\partial_s u(x, s)|^2 dx ds\|_{\mathcal{C}}, \end{aligned}$$

where  $\widehat{I} = (a, 2b - a)$  when  $I = (a, b)$  with  $0 \leq a < b < \infty$ .

Assume now that  $|I| < 1/k$ . According to (16) we deduce that

$$\left| \partial_t u\left(x, t + \frac{1}{k}\right) \right| \leq \frac{C}{t + 1/k} \|s |\partial_s u(x, s)|^2 dx ds\|_{\mathcal{C}}^{1/2}, \quad x, t \in (0, \infty).$$

Then

$$(42) \quad \begin{aligned} \frac{1}{|I|} \int_0^{|I|} \int_I t \left| \partial_t u\left(x, t + \frac{1}{k}\right) \right|^2 dx dt &\leq \frac{C}{|I|} \|s |\partial_s u(x, s)|^2 dx ds\|_{\mathcal{C}} \int_0^{|I|} \int_I \frac{t}{(t + 1/k)^2} dx dt \\ &\leq C \|s |\partial_s u(x, s)|^2 dx ds\|_{\mathcal{C}} k^2 \int_0^{|I|} t dt \leq C \|s |\partial_s u(x, s)|^2 dx ds\|_{\mathcal{C}}. \end{aligned}$$

Putting together (41) and (42) we prove (40).  $\square$

We define, for every  $k \in \mathbb{N}$ ,  $f_k(x) = u_k(x, 0)$ ,  $x \in (0, \infty)$ . By (32), (33), (40) and Lemma 3.1 we obtain that, for every  $k \in \mathbb{N}$ ,  $f_k \in BMO_o(\mathbb{R})$  and

$$(43) \quad \|f_k\|_{BMO_o(\mathbb{R})} \leq C \|s |\partial_s u(x, s)|^2 dx ds\|_{\mathcal{C}}^{1/2}.$$

Hardy spaces associated with Bessel operators have been studied in [7] and [24]. A function  $f \in L^1(0, \infty)$  is in the Hardy space  $H_o^1(\mathbb{R})$  provided that

$$\sup_{t>0} |P_t^\nu(f)| \in L^1(0, \infty),$$

for some (equivalently, for every)  $\nu > 1$ . For every  $\nu > 1$ , we define

$$\|f\|_{H_\nu^1} := \left\| \sup_{t>0} |P_t^\nu(f)| \right\|_{L^1(0,\infty)}, \quad f \in H_0^1(\mathbb{R}).$$

For each  $\nu, \mu > 1$ , the norms  $\|\cdot\|_{H_\nu^1}$  and  $\|\cdot\|_{H_\mu^1}$  are equivalent on  $H_0^1(\mathbb{R})$ . The space  $H_0^1(\mathbb{R})$  endowed with the norm  $\|\cdot\|_{H_\nu^1}$  ( $\nu > 1$ ) is a Banach space. The dual space of  $H_0^1(\mathbb{R})$  is  $BMO_o(\mathbb{R})$  ([20, Theorem 1]).

To finish the proof the following results will be useful.

**Lemma 3.6.** *Let  $\nu > 0$ . For every  $x, t \in (0, \infty)$ ,  $P_t^\nu(x, \cdot) \in H_0^1(\mathbb{R})$ .*

*Proof.* Let  $x, t \in (0, \infty)$ . From the semigroup property it follows that

$$P_s^\nu[P_t^\nu(x, \cdot)](z) = \int_0^\infty P_s^\nu(z, y) P_t^\nu(x, y) dy = P_{t+s}^\nu(x, z), \quad z, s \in (0, \infty).$$

According to [42, p. 86, (b)] we have that

$$\begin{aligned} |P_s^\nu[P_t^\nu(x, \cdot)](z)| &\leq C \frac{(t+s)(xz)^\nu}{((t+s)^2 + (x-z)^2)^{\nu+1}} \leq C \frac{(xz)^\nu}{(t+s+|x-z|)^{2\nu+1}} \\ &\leq C \frac{(xz)^\nu}{(t+|x-z|)^{2\nu+1}}, \quad z, s \in (0, \infty). \end{aligned}$$

Then,

$$\begin{aligned} \int_0^\infty \sup_{s>0} |P_s^\nu[P_t^\nu(x, \cdot)](z)| dz &\leq C \left( \int_0^{2x} + \int_{2x}^\infty \right) \frac{(xz)^\nu}{(t+|x-z|)^{2\nu+1}} dz \\ &\leq C \left( \int_0^{2x} \frac{(xz)^\nu}{t^{2\nu+1}} dz + \int_{2x}^\infty \frac{(xz)^\nu}{(t+z)^{2\nu+1}} dz \right) \leq C \left( \left(\frac{x}{t}\right)^{2\nu+1} + \left(\frac{x}{t}\right)^\nu \right). \end{aligned}$$

Thus, we prove that  $P_t^\nu(x, \cdot) \in H_0^1(\mathbb{R})$ .  $\square$

**Lemma 3.7.** *Assume that  $g \in BMO_o(\mathbb{R})$  and  $G \in H_0^1(\mathbb{R})$  satisfying that  $gG \in L^1(0, \infty)$ . Then,*

$$(44) \quad \langle g, G \rangle_{BMO_o(\mathbb{R}), H_0^1(\mathbb{R})} = \int_0^\infty g(x) G(x) dx$$

*Proof.* According to the atomic characterization of  $H_0^1(\mathbb{R})$  ([7, Theorem 1.10]) we can find a sequence of measurable functions of compact support  $(G_j)_{j \in \mathbb{N}}$  such that, for every  $j \in \mathbb{N}$ ,  $G_j$  is a linear combination of  $H_0^1(\mathbb{R})$ -atoms, and  $G_j \rightarrow G$ , as  $j \rightarrow \infty$ , in  $H_0^1(\mathbb{R})$ . Then,

$$\langle g, G \rangle_{BMO_o(\mathbb{R}), H_0^1(\mathbb{R})} = \lim_{j \rightarrow \infty} \langle g, G_j \rangle_{BMO_o(\mathbb{R}), H_0^1(\mathbb{R})} = \lim_{j \rightarrow \infty} \int_0^\infty g(x) G_j(x) dx.$$

On the other hand, since  $gG \in L^1(0, \infty)$  and  $gG_j \in L^1(0, \infty)$ ,  $j \in \mathbb{N}$ , by [5, p. 25], we have that

$$\left| \int_0^\infty g(x) (G(x) - G_j(x)) dx \right| \leq \|g\|_{BMO_o(\mathbb{R})} \|G - G_j\|_{H_0^1(\mathbb{R})}, \quad j \in \mathbb{N}.$$

By letting  $j \rightarrow \infty$ , we conclude (44).  $\square$

By using Banach-Alaoglu theorem and by taking into account (43) there exists  $f \in BMO_o(\mathbb{R})$  and a strictly increasing  $\phi : \mathbb{N} \rightarrow \mathbb{N}$  such that  $f_{\phi(k)} \rightarrow f$ , as  $k \rightarrow \infty$ , in the weak star topology of  $BMO_o(\mathbb{R})$ , that is, for every  $g \in H_0^1(\mathbb{R})$ ,

$$(45) \quad \langle f_{\phi(k)}, g \rangle_{BMO_o(\mathbb{R}), H_0^1(\mathbb{R})} \rightarrow \langle f, g \rangle_{BMO_o(\mathbb{R}), H_0^1(\mathbb{R})}, \quad \text{as } k \rightarrow \infty.$$

Moreover,

$$\|f\|_{BMO_o(\mathbb{R})} \leq C \|s |\partial_s u(x, s)|^2 dx ds\|_{\mathcal{C}}^{1/2}.$$

By using (45) and Lemma 3.6 we obtain, for every  $x, t \in (0, \infty)$ ,

$$\langle f_{\phi(k)}, P_t^\lambda(x, \cdot) \rangle_{BMO_o(\mathbb{R}), H_0^1(\mathbb{R})} \rightarrow \langle f, P_t^\lambda(x, \cdot) \rangle_{BMO_o(\mathbb{R}), H_0^1(\mathbb{R})}, \quad \text{as } k \rightarrow \infty.$$

Since  $BMO_o(\mathbb{R}) \subset BMO(\mathbb{R})$ , by [42, p. 86, (b)] and [50, p. 141], for every  $g \in BMO_o(\mathbb{R})$ , we get

$$\begin{aligned} \int_0^\infty |g(y)| |P_t^\lambda(x, y)| dy &\leq Ct \int_0^\infty \frac{|g(y)|}{1+y^2} dy \sup_{y \in (0, \infty)} \frac{1+y^2}{t^2 + (x-y)^2} \\ &\leq Ct \left( \frac{1}{t^2} + \sup_{y \in (0, \infty)} \frac{y^2}{t^2 + (x-y)^2} \right) \leq Ct \left( \frac{1+x^2}{t^2} + 1 \right), \quad x, t \in (0, \infty). \end{aligned}$$

For every  $x, t \in (0, \infty)$ , Lemma 3.7 leads to

$$\int_0^\infty f_{\phi(k)}(y) P_t^\lambda(x, y) dy \longrightarrow \int_0^\infty f(y) P_t^\lambda(x, y) dy, \quad \text{as } k \rightarrow \infty.$$

By (32) we conclude that

$$u(x, t) = \int_0^\infty f(y) P_t^\lambda(x, y) dy, \quad x, t \in (0, \infty).$$

Thus the proof is finished.

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